Math 314 Summer '11 Exam 2

Name: ______________________ **Score:** ______________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (20 pts) True/False. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.

   (a) T or F? The zero vector $\mathbf{0}$ is always an eigenvector of any matrix A.

   (b) T or F? It is always possible to find an orthogonal basis for an eigenspace E_λ.

   (c) T or F? The projection of a vector $\mathbf{v}$ to a subspace W is neither inside W nor $W^\perp$.

   (d) T or F? If λ and μ are eigenvalues of A and B, respectively, then $\lambda + \mu$ is an eigenvalue of $A + B$.

   (e) T or F? A matrix is diagonalizable if and only if it is invertible.

   (f) T or F? $T(x) = \pi^2 x$ is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

   (g) T or F? There are linear transformations $T : \mathbb{R}^n \to \mathbb{R}^m$ that are given by $T(\vec{x}) = \vec{b} + A\vec{x}$ for some nontrivial vector $\vec{b} \neq 0$ and matrix A.

   (h) T or F? An $n \times n$ matrix is diagonalizable if and only if it has n distinct eigenvalues.

   (i) T or F? A matrix is symmetric if and only if it is orthogonally diagonalizable.

   (j) T or F? If Q is an orthogonal matrix, then $(Q^T\mathbf{x}) \cdot (Q^T\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

2. (10 pts) (a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection to the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find its standard matrix $[T]$.

   (b) Use the standard matrix to find the vector $T \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

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3. (20 pts) Given the following similarity relations

$$Q^{-1}AQ = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -2 \\ 6 & -9 & 7 \end{bmatrix} \overbrace{\begin{bmatrix} -8 & 12 & -10 \\ -2 & 3 & -2 \\ 6 & -9 & 8 \end{bmatrix}}^{A} \begin{bmatrix} 3 & -2 & -1 \\ 2 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$R^{-1}BR = \begin{bmatrix} 1 & -1 & 1 \\ -2 & 3 & -2 \\ 5 & -8 & 6 \end{bmatrix} \overbrace{\begin{bmatrix} -8 & 14 & -10 \\ 2 & -2 & 2 \\ 11 & -17 & 13 \end{bmatrix}}^{B} \begin{bmatrix} 2 & -2 & -1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) What are the eigenvalues of A and their algebraic multiplicities?

(b) Find an eigenvector for the smallest eigenvalue of A.

(c) Find an invertible matrix P so that $P^{-1}AP = B$.

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4. (15 pts) Let $A = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

(a) Use a combination of row/column expansion to find the characteristic equation of A.

(b) Find an eigenvector for the smallest, real eigenvalue of A.

5. (10 pts) Use a combination of row/column expansions and elementary row operations to find the determinant $\det(A)$ of matrix $A = \begin{bmatrix} 1 & 3 & 2 & 4 & 3 \\ 0 & 4 & 2 & 4 & 3 \\ 2 & 9 & 7 & 12 & 9 \\ 1 & 5 & 4 & 8 & 6 \\ 1 & 4 & 3 & 6 & 6 \end{bmatrix}$.

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6. (25 pts) Let $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$ and $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the subspace $W^{\perp}$.

(b) Use the Gram-Schmidt process to find an orthogonal basis for W.

(c) Find $\mathbf{w}^{\perp}$ from $W^{\perp}$ and $\mathbf{w}$ from W so that $\mathbf{v} = \mathbf{w}^{\perp} + \mathbf{w}$.

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7. (20 pts) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(a) Find all eigenvalues of A.

(b) Find all eigenvectors of A.

(c) Find an orthogonal matrix Q and a diagonal matrix D so that $A = QDQ^T$.

8. (5 pts) Let A be an invertible and symmetric matrix. Prove that its inverse A^{-1} is also symmetric.

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2 Bonus Points: True of False: Pound for pound there are more ants than the total weight of all living people on Earth.

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