Solution Key Math 314 Final Exam Spring 2014

1 (20 pts) (a) F (b) T (c) F (d) T (e) F (f) T (g) F (h) F

2 (10 pts) $A = \begin{vmatrix} r & r & 0 \\ 0 & r(r-2) & 0 \\ 1 & r^3 & 3(1-r) \end{vmatrix} = r(r-2)\begin{vmatrix} r & 0 \\ 1 & 3(1-r) \end{vmatrix} = 3r^2(1-r)(r-2) \neq 0 \Rightarrow r \neq 0, 1, 2.$

3 (10 pts) $T(x_1, x_2) = (x_1 + 2x_2,\ x_2 - x_1,\ 2x_1) \Rightarrow [T] = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 0 \end{bmatrix}$

4 (35 pts) (a) $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\det(A) = 0$ since $\det(B) = 0$.

(c) $\begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -8 \\ 9 \\ 5 \\ 4 \\ -6 \end{bmatrix} = -\begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_5 \Rightarrow \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

5 (15 pts) (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{-R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{-R_3 + R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \xrightarrow{\frac{1}{2} R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3} R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$. (b) $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

6 (15 pts) (a) $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in W^\perp \Leftrightarrow x_1 + x_2 + x_3 = 0 \Rightarrow \vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} x_3$ $W^\perp = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) $v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $w = \operatorname{proj}_u v = \frac{v \cdot u}{u \cdot u} u = \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. $w^\perp = v - w = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

7 (20 pts) (a) $\begin{vmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$, $\begin{vmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{vmatrix} = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & -3 & 3 \\ 1 & -2 & 1 \\ 0 & 3 & -3 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{vmatrix} = 0$. Yes.

(b) $\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $E_{\lambda=1} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$\to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to E_{\lambda=4} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ (c) $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

8 (15 pts) (a) $P_{C \leftarrow B} = [[b_1]_C\ [b_2]_C] = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. (b) $[p]_C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $[p]_B = P_{B \leftarrow C}[p]_C$
$= (P_{C \leftarrow B})^{-1}[p]_C = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \end{bmatrix}$.

9 (15 pts) $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2$
$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \end{bmatrix}$ or $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$, $\Rightarrow W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

10 (15 pts) $\det|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$ $\lambda_1 = +1$, $\lambda_2 = -1$. $A - \lambda_1 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$A - \lambda_2 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $\Rightarrow U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$, $A = U \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U^T = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

11 (15 pts) $A^T A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 0$ $[A^T A - 0I] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,
$A^T A - \lambda_1 I = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \to v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $u_1 = \frac{1}{\|Av_1\|} Av_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A = U \Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$

12 (15 pts) $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} / \sqrt{3}$, $u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} / \sqrt{2}$, $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. $\Rightarrow A = \sqrt{3}\, u_1 v_1^T + 2\sqrt{2}\, u_2 v_2^T \Rightarrow \sigma_1 = 2\sqrt{2}$, $\sigma_2 = \sqrt{3}$.

Bonus 2 F