Math 314 Spring '14 Final Exam

Name: ______________________ **Score:** ______________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (20 pts) For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.

   (a) T or F? Every orthogonal matrix is symmetric.

   (b) T or F? The matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable.

   (c) T or F? If A is an $m \times n$ with $m < n$, then the dimension of its column space is greater than the dimension of its row space.

   (d) T or F? The null space null(A) of a matrix A is orthogonal to the column space of A^T.

   (e) T or F? Zero can be the eigenvalue of an elementary matrix.

   (f) T or F? If $A_{n\times n}$ is an orthogonal matrix then A must be invertible.

   (g) T or F? If $A_{n\times n}$ is invertible then A is diagonalizable.

   (h) T or F? The projection of any vector to another vector must not be a zero vector.

2. (10 pts) Find all real number r such that the matrixes $A = \begin{bmatrix} 1 & r & 0 \\ 0 & r^2 & 0 \\ 1 & r^3 & 3 \end{bmatrix} - \begin{bmatrix} r & 0 & 0 \\ 0 & 2r & 0 \\ 0 & 0 & 3r \end{bmatrix}$ is invertible.

3. (10 pts) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ with $T(x_1, x_2) = (x_1+2x_2, x_2-x_1, 2x_1)$. Find its matrix of transformation $[T]$.

4. (35 pts) The following matrices are row equivalent

$$A = \begin{bmatrix} 3 & -1 & 7 & -1 & -8 \\ 3 & 2 & 4 & 3 & 9 \\ 2 & 1 & 3 & 2 & 5 \\ 1 & 1 & 1 & 1 & 4 \\ 1 & -1 & 3 & -1 & -6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the reduced row echelon form, $rref(A)$, of A.

(b) Find the determinant of A. Justify your answer without using calculator. (*Hint*: If you are not using the information given you may have to spend value time on this problem.)

(c) Find a basis for the column space of A.

(d) Write the fifth column of A as a linear combination of the other columns.

(e) Find a basis for the nullspace of A.

5. (15 pts) (a) Find the inverse matrix A^{-1} of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$.

(b) Write A^{-1} as a product of elementary matrices.

6. (15 pts) Let $W = \text{span}\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \}$.

(a) Find a base for $W^{\perp}$.

(b) Decompose $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ as $\mathbf{v} = \mathbf{w} + \mathbf{w}^{\perp}$ with $\mathbf{w}$ from W and $\mathbf{w}^{\perp}$ from $W^{\perp}$.

7. (20 pts) (a) Verify that $\lambda = 1, 4$ are eigenvalues of $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(b) Find a basis for the eigenspace E_λ for each eigenvalue.

(c) Diagonalize the matrix A by finding P and D to factorize A as $A = PDP^{-1}$. (No need to find P^{-1}.)

8. (15 pts) Let $\mathcal{B} = \{2 + t, 5 + 3t\}$ and $\mathcal{C} = \{1, t\}$ be two bases of $\mathbb{P}_2$.

(a) Find the change of basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$

(b) If $\mathbf{p}(t) = 1 + 2t$, find both $[\mathbf{p}]_{\mathcal{C}}$ and $[\mathbf{p}]_{\mathcal{B}}$.

9. (15 pts) Let $W = \text{span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

10. (15 pts) Find a spectral decomposition of $A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

11. (15 pts) Find a singular value decomposition of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

12. (15 pts) A matrix is given as $A = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} [2, 0] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [0, 1]$. Show that the singular values of A are $\sigma_1 = 2\sqrt{2},\ \sigma_2 = \sqrt{3}$. (*Hint:* You can use a shortcut to justify your answer without finding A.)

2 Bonus Points: True or False: The Nebraska State Book is the Bible.