Math 314 Fall '10 Exam 1

Name: ______________________ **Score:** ______________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (20 pts) True/False. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.
   (Recall: an $m \times n$ matrix is one that has m rows and n columns.)

   (a) T or F? Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 1$.

   (b) T or F? Both row(A) and col(A) are preserved by the elementary row operations on A.

   (c) T or F? For matrix multiplication, if $AB = 0$, then either $A = 0$ or $B = 0$.

   (d) T or F? The dimension of span$(\{\mathbf{v}_1, ..., \mathbf{v}_n\})$ is always n.

   (e) T or F? The trivial solution $\mathbf{x} = 0$ is the only solution for every homogeneous equation $A\mathbf{x} = 0$.

   (f) T or F? The nullity(A) is equal to the dimension of the solution set for $A\mathbf{x} = \mathbf{0}$.

   (g) T or F? The rank(A) is equal to the rank(A^T).

   (h) T or F? A square matrix A is invertible if nullity$(A) \leq$ rank(A).

   (i) T or F? Every system of two equations in three unknowns has infinitely many solutions.

   (j) T or F? The unit disk $S = \{(x, y) : \ x^2 + y^2 \leq 1\}$ is a linear subspace of $\mathbb{R}^2$.

2. (10 pts) Suppose $AB\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find $\mathbf{x}$ if you know that $A^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(*Continue on Next Page ...*)

3. (10 pts) Find the inverse of the given matrix using the Gauss-Jordan method:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

4. (10 pts) Find the LU-factorization of this matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

That is, find a lower triangular matrix L with all diagonal entries being 1 and an upper triangular matrix U so that $A = LU$.

(*Continue on Next Page ...*)

5. (20 pts) The following matrices (of dimensions 5×6) are row-equivalent:

$$A = \begin{bmatrix} 2 & 0 & -2 & 5 & 0 & 6 \\ 1 & 8 & 3 & -2 & 4 & -9 \\ 2 & 2 & -1 & -1 & 1 & 3 \\ 3 & 4 & -1 & -1 & 2 & 3 \\ -2 & 2 & 3 & 7 & 1 & -9 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 3 \\ 0 & 1 & 1/2 & 0 & 1/2 & -3/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Answer the following questions and **briefly** justify your reasoning:

(a) What is the rank of A? ____

(b) What is the nullity of A? ____

(c) Are the first three columns of A linearly independent? Why or why not? If not, find a linear combination that shows the dependence relation of three vectors.

(d) Find a basis for the row space, row(A), of A.

(e) Find a basis for the null space of A.

(*Continue on Next Page ...*)

6. (15 pts) Let $S = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ with $\mathbf{u}_1 = \begin{bmatrix} 16 \\ 5 \\ 9 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 11 \\ 7 \\ 14 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 3 \\ 10 \\ 6 \\ 15 \end{bmatrix}$. Let $\mathbf{w} = \begin{bmatrix} 13 \\ 8 \\ 12 \\ 1 \end{bmatrix}$. You can use the following fact

$$\text{rref}\left(\begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\text{rref}(A)$ is defined to be the *reduced row echelon form* of a matrix A.

(a) Is $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ a basis of S? Why or why not?

(b) Is $\mathbf{w}$ in the span S? Why or why not?

(c) If yes to both questions above, what are the coordinates of $\mathbf{w}$ with respect to the basis, i.e. $[\mathbf{w}]_{\mathcal{B}}$? If no to either question, find a set of general equations for the span.

(d) Suppose the coordinates of some vector $\mathbf{v}$ with respect to $\mathcal{B}$ are $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. What is $\mathbf{v}$ in the standard coordinates?

(*Continue on Next Page ...*)

7. (5 pts) If a square matrix A satisfies the equation $A^2 - A + I = 0$, show that it must be invertible, and find its inverse A^{-1}.

8. (5 pts) Assume the product AB makes sense. Prove that if the rows of A are linearly dependent so are the rows of AB.

9. (5 pts) If A is invertible, prove that its transpose A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

---

2 Bonus Points: The state bird of Nebraska is (a) The Northern Cardinal , (b) The Blue Jay, (c) The Western Meadowlark, (d) none of the above. (... *The End*)