Math 314 Summer '11 Final Exam

Name: ______________________________ **Score:** ______________________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (20 pts) True/False. For each of the following statements, please *circle* T (True) or F (False). You do not need to justify your answer.

   (a) T or F? If A is a 5×8 matrix then the nullity of A is at least 3.

   (b) T or F? The columns of any 3×4 matrix can be linearly independent.

   (c) T or F? If W is a vector space spanned by 4 vectors, them the dimension of W is 4.

   (d) T or F? If A and B are $n \times n$ matrices such $AB = 0$ then $BA = 0$.

   (e) T or F? The elementary row operations preserve the eigenvalues of any matrix.

   (f) T or F? If A is an $m \times n$ with $m < n$, then the dimension of its row space is also less than the dimension of the column space.

   (g) T or F? If there are infinitely many solutions to a nonhomogeneous equation $A\vec{x} = \vec{b}$, then the dimension of nullspace null(A) must be zero.

   (h) T or F? If A is an orthogonal matrix, then $\det(A) = 1$.

   (i) T or F? $W = \{[x, y, z]^T : x + 2y + 3z = 4\}$ is a linear subspace of $\mathbb{R}^3$.

   (j) T or F? Let $W = \text{span}\{\vec{v}_1, \dots, \vec{v}_k\}$ be a subspace of $\mathbb{R}^n$, $\vec{x} = x_1\vec{v}_1 + \cdots + x_k\vec{v}_k$ and $\vec{y} = y_1\vec{v}_1 + \cdots + y_k\vec{v}_k$. Then $\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_ky_k$.

2. (10 pts) Find all real number k such that the determinant of the matrix $A = \begin{bmatrix} 1 & 4 & 3 & 1 \\ k & 1 & 0 & 3 \\ 1 & 2 & 3 & 1 \\ 3 & 5 & 0 & 1 \end{bmatrix}$ is not zero.

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3. (30 pts) The following matrices are row equivalent

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 & 2 & 0 & -2 \\ 0 & 2 & -2 & 1 & 5 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 0 & -2 & 0 & 2 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 & -1 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find the rank of A and the nullity of A.

(b) Find a basis, $\mathcal{B}$, for the column space of A.

(c) Is the vector $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ in the column space of A? If yes, find the coordinate $[\mathbf{b}]_{\mathcal{B}}$.

(d) Find a basis for the nullspace of A.

(e) Find a basis for the subspace W that is orthogonal to the nullspace, null(A), of A.

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4. (15 pts) (a) Find the inverse matrix A^{-1} of $A = \begin{bmatrix} -3 & -2 & -1 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$.

(b) Write A as a product of elementary matrices.

5. (15 pts) Let $W = \{ \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} : w + x = 0, x + y + z = 0 \}$.

(a) Verify that W is a linear subspace of $\mathbb{R}^4$.

(b) Find a basis for $W^{\perp}$.

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6. (15 pts) Let T be a linear transformation in $\mathbb{R}^2$ satisfying $T(\mathbf{e}_1) = 2\mathbf{e}_1 + \mathbf{e}_2$ and $T(\mathbf{e}_2) = 3\mathbf{e}_1 + 4\mathbf{e}_2$.

(a) Find its matrix of transformation $[T]$.

(b) Let S be another linear transformation with $[S] = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. Find $T \circ S(\begin{bmatrix} -1 \\ 1 \end{bmatrix})$.

7. (15 pts) Let $A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 4 & 5 \end{bmatrix}$.

(a) Verify that $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ is an eigenvector of A and determine corresponding eigenvalue λ.

(b) Find a basis for the eigenspace E_λ for the eigenvalue λ from (a).

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8. (15 pts) Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be two bases of $\mathbb{R}^2$.

(a) Find the change of basis matrix $P_{\mathcal{C} \leftarrow \mathcal{B}}$ from $\mathcal{B}$ to $\mathcal{C}$

(b) If $\mathbf{v} = \mathbf{b_1} + 2\mathbf{b}_2$, find both $[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{C}}$.

9. (15 pts) Let $W = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Use the Gram-Schmidt process to find an orthogonal basis for W.

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10. (15 pts) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. Find an outer product form of the singular value decomposition (SVD) for A.

11. (15 pts) For the bilinear form $f(\mathbf{x}) = x_1^2 + x_2^2 + 2x_1x_2$, find an orthogonal change of variables $\mathbf{x} = Q\mathbf{y}$ with $Q^T = Q$ so that in the new variable $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ the bilinear form becomes $f(\mathbf{x}) = ay_1^2 + by_2^2$ for some real numbers a, b.

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12. (10 pts) If a symmetric matrix A has eigenvalues $\lambda_1 = 0, \lambda_2 = 1$ with an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$ for λ_1, find an orthogonal matrix Q and a diagonal matrix D so that $Q^T AQ = D$.

13. (5 pts) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis of $\mathbb{R}^n$. Prove that for any two vectors $\mathbf{x}, \mathbf{y}$ from $\mathbb{R}^n$, $[\mathbf{x} + \mathbf{y}]_\mathcal{B} = [\mathbf{x}]_\mathcal{B} + [\mathbf{y}]_\mathcal{B}$.

14. (5 pts) Let A be an $n \times n$ symmetric matrix, $A^T = A$, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A. Show that $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$ are the singular values of A.

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2 Bonus Points: The number of times that homework are collected are (a) 16, (b) 18, (c) 20, (d) 22.

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