Name:____________________________ Score:________

Instructions: You must show supporting work to receive full and partial credits. No text book, personal notes, formula sheets allowed. One formula sheet will be provided.

1(**10pts**) Use definition *only* to find the Laplace transform of the function

$$f(t) = \begin{cases} 2, & t < b \\ 0, & t \geq b \end{cases}$$

Using any other method receives no credit.

2(**25pts**) Find the Laplace transforms of these functions

(a) $(t+2)(t+e^{2t})$

(b) $f(t) = \begin{cases} t, & t < 2 \\ 0, & 2 \leq t < 4, \\ 4, & t \geq 4 \end{cases}$ which must be first expressed as a linear combination of various unit step functions.

(c) $t \cos 2t$

3(**25pts**) Find the Laplace inverses of these functions

(a) $\dfrac{3}{s^2+s-2}$

(b) $\dfrac{s+e^{-3s}}{s^2+4s+5}$

(c) $\dfrac{3s^2+5s+6}{(s+1)(2s^2+4s+6)}$

4(**15pts**) Fill in the following blanks and justify your answers:

(a) If $\mathcal{L}\{f(t)\}(s) = \dfrac{1}{\sqrt{s}}$, then $\mathcal{L}\{t^2 f(t)\} =$ __________.

(b) If $\mathcal{L}^{-1}\{F(s)\}(t) = \sqrt{t}$, then $\mathcal{L}^{-1}\{e^{-2s}(F(s)+1)\}(t) =$ __________.

5(**15pts**) Solve the initial value problem: $\begin{cases} x'' + 2x' + x = u_3(t) \\ x(0) = 0,\ x'(0) = 0. \end{cases}$

6(**10pts**) Use the Laplace transformation method to find $\mathcal{L}\{x(t)\}(s), \mathcal{L}\{y(t)\}(s)$ for this system of equations:

$$\begin{cases} x' = x + t \\ y' = x + y \\ x(0) = 0,\ y(0) = 0. \end{cases}$$

(You are not asked to find the solution $x(t), y(t)$.)

Bonus 3pts: True or false: A C- grade for this class is considered a pass. ______

The End

Soln Key Test 5 Math221 Summer(2), '02

1 (10pts) $\mathcal{L}\{f(t)\}(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^b 2e^{-st}dt = -\frac{2}{s}e^{-st}\Big|_0^b = \boxed{\frac{2}{s}(1-e^{-bs})}$

2 (25pts) (a) $(t+2)(t+e^{2t}) = t^2 + te^{2t} + 2t + 2e^{2t} \xrightarrow{\mathcal{L}} \boxed{\frac{2}{s^3} + \frac{1}{(s-2)^2} + \frac{2}{s} + \frac{2}{s-2}}$

(b) $f(t) = \begin{cases} t, & t<2 \\ 0, & 2\le t<4 \\ 4, & t\ge 4 \end{cases} = t - tu_2(t) + 4u_4(t) \xrightarrow{\mathcal{L}} \boxed{\frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) + 4e^{-4s}\frac{1}{s}}$

(c) $t\cos t \xrightarrow{\mathcal{L}} -\frac{d}{ds}\frac{s}{s^2+1} = -\frac{s^2+1-2s^2}{(s^2+1)^2} = \boxed{\frac{s^2-1}{(s^2+1)^2}}$

3 (25pts) (a) $\frac{3}{s^2+s-2} = \frac{3}{(s+2)(s-1)} = \frac{1}{s-1} - \frac{1}{s+2} \xrightarrow{\mathcal{L}^{-1}} \boxed{e^t - e^{-2t}}$ $\left(\frac{3}{(s+2)(s-1)} = \frac{A}{s-1} + \frac{B}{s+2}\right.$ solve for $A, B = 1, 1 \left.\right)$

(b) $\frac{s+e^{-3s}}{s^2+4s+5} = \frac{s}{(s+2)^2+1} + \frac{e^{-3s}}{(s+2)^2+1} = \frac{s+2-2}{(s+2)^2+1} + \frac{e^{-3s}}{(s+2)^2+1}$

$= \frac{s+2}{(s+2)^2+1} - 2\frac{1}{(s+2)^2+1} + e^{-3s}\frac{1}{(s+2)^2+1} \xrightarrow{\mathcal{L}^{-1}} \boxed{e^{-2t}\cos t - 2e^{-2t}\sin t + u_3(t)\, e^{-2(t-3)}\sin(t-3)}$

(c) $\frac{3s^2+5s+6}{(s+1)(2s^2+4s+6)} = \frac{A}{s+1} + \frac{Bs+C}{2s^2+4s+6}$ (since $2s^2+4s+6$ is irreducible for $4^2-4(2)(6)<0$) $= \frac{A(2s^2+4s+6)+(Bs+C)(s+1)}{(s+1)(2s^2+4s+6)}$

$\Rightarrow$ s^2: $2A+B=3$ ①
s: $4A+B+C=5$ ②
1: $6A+C=6$ ③

$\xrightarrow{②-①} \begin{cases} 2A+C=2 & ① \\ 6A+C=6 & ② \end{cases}$ $\xrightarrow{②-①}$ $4A=4$, $A=1$, $B=1$, $C=0$

$\Rightarrow \frac{3s^2+5s+6}{(s+1)(2s^2+4s+6)} = \frac{1}{s+1} + \frac{1}{2}\frac{1}{s^2+2s+3} = \frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2+2} = \frac{1}{s+1} + \frac{1}{2\sqrt{2}}\frac{\sqrt{2}}{(s+1)^2+2} \xrightarrow{\mathcal{L}^{-1}} \boxed{e^{-t} + \frac{1}{2\sqrt{2}}e^{-t}\sin\sqrt{2}t}$

4 (15pts) (a) $\mathcal{L}\{f\} = \frac{1}{\sqrt{s}}$, $\mathcal{L}\{t^2f(t)\}(s) = (-1)^2\frac{d^2}{ds^2}\mathcal{L}\{f\} = \frac{d^2}{ds^2}s^{-\frac{1}{2}} = \left(-\frac{1}{2}\right)\frac{d}{ds}s^{-\frac{3}{2}} = \boxed{\frac{3}{4}s^{-\frac{5}{2}}}$

(b) $\mathcal{L}^{-1}\{F(s)\} = \sqrt{t}$, $\mathcal{L}^{-1}\{e^{-2s}(F(s)+1)\}(t) = \mathcal{L}^{-1}\{e^{-2s}F(s)\}(t) + \mathcal{L}^{-1}\{e^{-2s}\}(t)$

$= u_2(t)\mathcal{L}^{-1}\{F(s)\}(t-2) + \delta_2(t) = \boxed{u_2(t)\sqrt{t-2} + \delta_2(t)}$

5 (15pts) $\begin{cases} x''+2x'+x = u_3(t) \\ x(0)=x'(0)=0. \end{cases}$ $\mathcal{L}\{x''\} + 2\mathcal{L}\{x'\} + \mathcal{L}\{x\} = s^2\mathcal{L}\{x\} - sx(0) - x'(0) + 2(s\mathcal{L}\{x\} - x(0)) + \mathcal{L}\{x\} = (s^2+2s+1)\mathcal{L}\{x\} = \mathcal{L}\{u_3(t)\} = \frac{1}{s}e^{-3s}$

$\Rightarrow \mathcal{L}\{x\} = e^{-3s}\frac{1}{s(s^2+2s+1)} = e^{-3s}\frac{1}{s(s+1)^2}$ $\Big|$ $\frac{1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{A(s^2+2s+1)+Bs(s+1)+Cs}{s(s+1)^2}$

s^2: $A+B=0$
s: $2A+B+C=0$ $\Rightarrow B=-1$, $C=-1$.
$A=1$

$x(t) = \mathcal{L}^{-1}\left\{e^{-3s}\frac{1}{s(s+1)^2}\right\} = u_3(t)\left(\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right\}\Big|_{t\to t-3}\right) = u_3(t)\left((1-e^{-t}-te^{-t})\Big|_{t\to t-3}\right)$

$= \boxed{u_3(t)\left(1-e^{-(t-3)} - (t-3)e^{-(t-3)}\right)}$

6 (10pts) $\begin{cases} x'=x+t \\ y'=x+y \\ x(0)=y(0)=0. \end{cases}$ $\mathcal{L}\{x'\} = s\mathcal{L}\{x\} - x(0) = s\mathcal{L}\{x\} = \mathcal{L}\{x\} + \mathcal{L}\{t\} = \mathcal{L}\{x\} + \frac{1}{s^2}$.

$\Rightarrow \boxed{\mathcal{L}\{x\} = \frac{1}{s^2(s-1)}}$

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0) = s\mathcal{L}\{y\} = \mathcal{L}\{x\} + \mathcal{L}\{y\} \Rightarrow \boxed{\mathcal{L}\{y\} = \frac{1}{(s-1)}\mathcal{L}\{x\} = \frac{1}{s^2(s-1)^2}}$

Bonus 3pts. False or whatever.