Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(20pts) Consider the system of equations $x' = Ax$ with $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$

(a) Find a solution general solution of the system.
(b) Find the solution satisfying the initial condition $x_1(0) = 1, x_2(0) = 2$.
(c) Sketch a phase portrait of the system, including all straight line solutions, and in particular the solution of (b) above.

2(15pts) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

3(15pts) It is given that $\lambda = -2 - 3i$ is an eigenvalue of a real valued $3 \times 3$ matrix $A$ and $u = \begin{pmatrix} 0 \\ i \\ -2 - 3i \end{pmatrix}$ is a corresponding eigenvector.

(a) Find two linearly independent solutions to $x' = Ax$.
(b) If $\begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$ is another solution, find a general solution.

4(20pts) Use phase plane method to carefully sketch a phase portrait of the system of equations
\[
\begin{align*}
x' &= y - x \\
y' &= x - y^2
\end{align*}
\]
in the first quadrant $x \geq 0, y \geq 0$ only, by including its nullclines, typical vector fields on and off the nullclines, separatrix if any, and a few typical solution curves.

5(15pts) Consider a cooperative system of two species
\[
\begin{align*}
x' &= x(1 - x + y) \\
y' &= y(1 - y + 2x)
\end{align*}
\]
(a) For $x(0) > 0, y(0) > 0$, will the solution converge to a co-existence state? If not where is it going? (Use a phase plane analysis to answer this question. Include a sufficient amount of your reasoning.)
(b) Based on your analysis above, is this a reasonable model for cooperative systems. Explain.

6(15pts) Consider a competitive system of two species
\[
\begin{align*}
x' &= x(1 - x - y) \\
y' &= y(1 - 2y - ax)
\end{align*}
\]
with a parameter $a > 0$.
(a) For what values of $a$ do the two species co-exist? (Use a phase plane analysis to answer this question. Include a sufficient amount of your reasoning.)
(b) If your co-existence condition found above in (a) is not satisfied, which species will die out? Again base your conclusion on a phase plane analysis.

Bonus 3pts: My house’s street number is 2821. Can I water my lawn legally today? ____________
Math 221 Test 4, Summer 2002

1. (20 pts) \[ x' = \left[ \begin{array}{c} -2 \\ 1 \end{array} \right] x. \] (c) \( |A-T| = \left| \begin{array}{cc} -2-T & 1 \\ -1 & -2-T \end{array} \right| = (-2-T)^2 - 1 = (T+1)^2 = 0 \] \( r = 0, -1 \)

2. (15 pts) \( (A-TI)x = 0 \) implies \( u_1 = 2u_2 \) \( \Rightarrow u_2 = u_1, \) \( \xi_1 = (1,0) \) \( \xi_2 = (0,1) \) \( U = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \)

3. (15 pts) \( e^{-2t+3it} = e^{-2t} (\cos(3t) + i\sin(3t)) \)

4. (20 pts) \( x' = y-x \) \( x'-y = 0 \) \( y'=y-x \) \( x-y = 0, x=y \)

5. (15 pts) \( x' = x(-x+y) \) \( y' = y(-y+2x) \)

6. (15 pts) \( x' = x(-x+y), \) \( y' = y(-y+2x) \)

Bonus 3 pts. (Yes) accept any answer though.