Aug. 9, 2002 **Math 221 Test 4** **Summer(2) 2002**

Name:______________________________ Score:__________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(**20pts**) Consider the system of equations $\mathbf{x}' = A\mathbf{x}$ with $A = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$

(a) Find a solution general solution of the system.

(b) Find the solution satisfying the initial condition $x_1(0) = 1, x_2(0) = 2$.

(c) Sketch a phase portrait of the system, including all straight line solutions, and in particular the solution of (b) above.

2(**15pts**) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

3(**15pts**) It is given that $\lambda = -2 - 3i$ is an eigenvalue of a real valued 3×3 matrix A and $\mathbf{u} = \begin{pmatrix} 0 \\ i \\ -2-3i \end{pmatrix}$ is a corresponding eigenvector.

(a) Find two linearly independent solutions to $\mathbf{x}' = A\mathbf{x}$.

(b) If $\begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$ is another solution, find a general solution.

4(**20pts**) Use phase plane method to carefully sketch a phase portrait of the system of equations

$$\begin{cases} x' = y - x \\ y' = x - y^2 \end{cases}$$

in the first quadrant $x \geq 0, y \geq 0$ only, by including its nullclines, typical vector fields on and off the nullclines, separatrix if any, and a few typical solution curves.

5(**15pts**) Consider a cooperative system of two species

$$\begin{cases} x' = x(1 - x + y) \\ y' = y(1 - y + 2x) \end{cases}$$

(a) For $x(0) > 0, y(0) > 0$, will the solution converge to a co-existence state? If not where is it going? (Use a phase plane analysis to answer this question. Include a sufficient amount of your reasoning.)

(b) Based on your analysis above, is this a reasonable model for cooperative systems. Explain.

6(**15pts**) Consider a competitive system of two species

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(1 - 2y - ax) \end{cases}$$

with a parameter $a > 0$.

(a) For what values of a do the two species co-exist? (Use a phase plane analysis to answer this question. Include a sufficient amount of your reasoning.)

(b) If your co-existence condition found above in (a) is not satisfied, which species will die out? Again base your conclusion on a phase plane analysis.

Bonus 3pts: My house's street number is 2821. Can I water my lawn legally today? ____________

The End

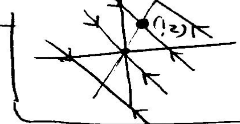
Math221 Test4 Soln Key Summer(2) 02

1 (20pts): $x' = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} x$. (a) $|A - rI| = \begin{vmatrix} -2-r & 1 \\ 4 & -2-r \end{vmatrix} = (2+r)^2 - 4 = 0$ $r_{1,2} = 0, -4$

$r_1 = 0$. $\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_2 = 2u_1$ $\vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. $r_2 = -4$, $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow u_2 = -2u_1$ $\vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

general soln: (a) $\vec{x}(t) = c_1\vec{\xi}_1 e^{r_1 t} + c_2\vec{\xi}_2 e^{r_2 t} = c_1\vec{\xi}_1 + c_2\vec{\xi}_2 e^{-4t}$. (b) $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{0}$

$\Rightarrow c_1 = 1, c_2 = 0$.



2 (15pts) $|A - rI| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 2 + 3\lambda$. $\lambda_1 = -1$. $(\lambda+1) \overline{) -\lambda^3 + 3\lambda + 2}$ quotient $-\lambda^2 + \lambda + 2$; $-\lambda^3 - \lambda^2$; $\lambda^2 + 3\lambda + 2$; $\lambda^2 + \lambda$; $2\lambda + 2$

$= (\lambda+1)(\lambda+1)(-\lambda+2) \Rightarrow \lambda_{1,2} = -1, \lambda_3 = 2$ $= (\lambda+1)(-\lambda^2 + \lambda + 2)$

$\lambda_{1,2} = -1$. $\Rightarrow (A - r_1 I)\vec{\xi} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \vec{0}$. $\Rightarrow u_1 + u_2 + u_3 = 0$ $u_1 = -(u_2 + u_3)$. $\Rightarrow \vec{\xi}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\xi}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = 2$ $(A - r_3 I)\vec{\xi} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $\begin{cases} -2u_1 + u_2 + u_3 = 0 & ① \\ u_1 - 2u_2 + u_3 = 0 & ② \\ u_1 + u_2 - 2u_3 = 0 & ③ \end{cases}$ ②+③ $\Rightarrow 2u_1 - u_1 - u_3 = 0$ same as ①

so only need $\begin{cases} -2u_1 + u_2 + u_3 = 0 & ① \\ u_1 - 2u_2 + u_3 = 0 & ② \end{cases}$ ①−② $\to -3u_1 - 3u_2 = 0$

$\Rightarrow u_2 = -u_1$ and $u_3 = -(2u_1 + u_2) = 3u_1 \Rightarrow \xi = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

3 (a) (15pts) $\begin{pmatrix} 0 \\ i \\ -2-3i \end{pmatrix} e^{-2t-3ti} = \begin{pmatrix} 0 \\ i \\ -2-3i \end{pmatrix} e^{-2t}(\cos 3t - i\sin 3t) = \begin{pmatrix} 0 \\ e^{-2t}\sin 3t + ie^{-2t}\cos 3t \\ e^{-2t}(-2\cos 3t - 3\sin 3t) + ie^{-2t}(-3\cos 3t + 2\sin 3t) \end{pmatrix}$

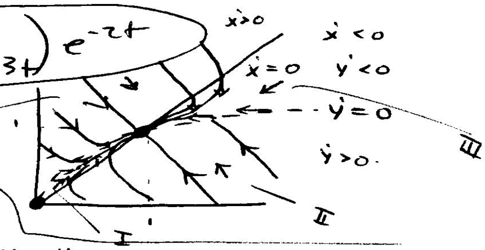
$\vec{x}_1(t) = \begin{pmatrix} 0 \\ \sin 3t \\ -2\cos 3t - 3\sin 3t \end{pmatrix} e^{-2t}$, $\vec{x}_2(t) = \begin{pmatrix} 0 \\ \cos 3t \\ 2\sin 3t - 3\cos 3t \end{pmatrix} e^{-2t}$

(b) $\vec{x}(t) = c_1\begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ \sin 3t \\ -2\cos 3t - 3\sin 3t \end{pmatrix} e^{-2t} + c_3\begin{pmatrix} 0 \\ \cos 3t \\ 2\sin 3t - 3\cos 3t \end{pmatrix} e^{-2t}$

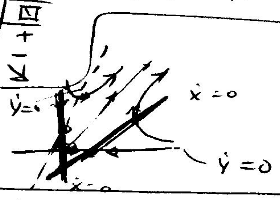
4 (20pts). $x' = y - x$ x-nullclines $y - x = 0, y = x$

$y' = x - y^2$ y-nullclines $x - y^2 = 0, x = y^2$

	I	II	III	IV
$\dot{x}$	+	−	−	+
$\dot{y}$	+	+	−	−
	↗	↖	↙	↘



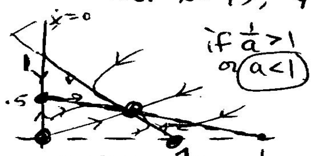
5 (15pts) $x' = x(1 - x + y)$
$y' = y(1 - y + 2x)$



(a) No. $x(t), y(t) \to \infty$ as $t \to \infty$.

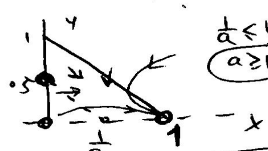
(b) Not, if x, y are limited by some carrying capacity.

6 (15pts) $x' = x(1 - x - y)$, $y' = y(1 - 2y - ax)$.



co-existing if $a < 1$

(a)



x dominant, y dies out if $x(0) > 0$; $a \ge 1$

(b)

Bonus 3pts. Yes acceptany answer though.