July 26, 2002 **Math 221 Test 2** **Summer 2002**

Name:______________________________ Score:________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(20pts) Consider the equation $tx'' - (t+2)x' + 2x = 0$.

(a) Verify that $x_1(t) = e^t$ and $x_2(t) = t^2 + 2t + 2$ are solutions to the equation for $t > 0$.

(b) Show that $x_1(t), x_2(t)$ form a fundamental set of solutions.

(c) Find the solution satisfying the initial conditions $x(1) = 0, x'(1) = 1$.

2(10pts) Determine the time interval in which the Existence and Uniqueness Theorem applies to this initial value problem $t(t-3)x'' + 2tx' - x = t^2; x(1) = x_0, x'(1) = x_1$.

3(10pts) An object of mass 100 kg falls off a platform 50 meters above the ground. Assume the force due to air resistance is negligible. Find the time when the object strikes the ground and the impact velocity.

4(15pts) Find a particular (synchronous) solution to the equation $x'' + x' + x = 2\sin(2t)$.

5(15pts) A brine solution of salt flows at a constant rate of 6L/min into a large tank that initially held 100L of pure water. The solution inside the tank is kept well stirred and flows out at a rate of 4L/min. If the tank has a capacity of holding 200L of mixture solution, when the tank will become overflow? Assume the concentration of salt in the brine entering the tank is 2kg/L. Derive an initial value problem for the amount of salt inside the tank during the time before overflowing occurs. (Do not solve the IVP.)

6(10pts) A mass of 2kg is attached to a spring which exerts an restoring force of 0.1N per cm. Assume the force due to friction is proportional to the velocity of the mass with the proportionality equal to 3N-sec/m.

(a) What is the spring constant according to Hooke's law?

(b) If the mass is at rest and is given an initial velocity of 0.5m/sec, write an initial value problem for the motion of the mass. (Do not solve the IVP.)

7(20pts) Bill turned off his apartment's air conditioner when he left for work in the morning. It took 10 minutes for the temperature in the apartment to arise from 78°F to 80°F while the temperature outside the apartment is 85°F. When he returned after work, his apartment was as hot as the outside at a temperature of 100°F. He turned on the air conditioner and set it to a cool 78°F again. Assume the air conditioner cills the air at a rate of 0.5°F per minute.

(a) What is Bill's apartment's proportionality constant (i.e., the reciprocal of the *time constant*) for Newton's law of cooling?

(b) Set up an initial value problem for the temperature after Bill turned on the air conditioner after his return from work.

(c) How long would it take the air conditioner to turn off automatically at 78°F?

END

Test 2 Math 221 Sdu Key Summer (2) '02

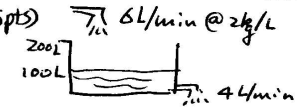
1 (20pts) $tx'' - (t+2)x' + 2x = 0$ (a) $x_1(t) = e^t$, $tx_1'' - (t+2)x_1' + 2x_1$
$= te^t - (t+2)e^t + 2e^t \equiv 0$. $x_2(t) = t^2+2t+2$, $x_2'' = 2$, $x_2' = 2t+2$, $tx_2'' - (t+2)x_2' + 2x_2$
$= t(2) - (t+2)2(t+1) + 2(t^2+2t+2) = 0t^2 + 0t + 0 \equiv 0 \Rightarrow x_1, x_2$ are solutions.
(b) $W[x_1, x_2] = \begin{vmatrix} e^t & t^2+2t+2 \\ e^t & 2t+2 \end{vmatrix} = e^t[2t+2 - t^2 - 2t - 2] = t^2e^t \neq 0$ for $t>0$.
$\Rightarrow x_1, x_2$ form a fundamental set. (c) $x(t) = c_1x_1 + c_2x_2 = c_1e^t + c_2(t^2+2t+2)$
is a general sdu. IC: $0 = x(1) = c_1e + 5c_2$, ① $1 = x'(1) = c_1e + 4c_2$ ②. ① − ② :⇒
$-1 = c_2 \Rightarrow c_1 = -5c_2/e = 5/e$. $\Rightarrow x(t) = 5e^{t-1} - t^2 - 2t - 2$

2 (10pts) $t(t-3)x'' + 2tx' - x = t^2$, $p(t) = \frac{2}{t-3}$ cont. in $(-\infty, 3), (3, \infty)$. $q(t) = \frac{-1}{t(t-3)}$
contin. in $(-\infty, 0), (0, 3), (3, \infty)$. $g(t) = \frac{t}{t-3}$ cont. in $(-\infty, 3), (3, \infty)$. ⇒ Thm applies
to the IVP in this interval $(0, 3)$.

3 (10pts) $100\frac{dv}{dt} = -9.81(100)$, $v(0) = 0$, $v' = -9.81$ $v(t) = -9.81t$. $x'(t) = v(t) = -9.81t$
$x(0) = 50$. $\Rightarrow x(t) = -\frac{9.81}{2}t^2 + 50$. Time of impact: $0 = x(t) = -\frac{9.81}{2}t^2 + 50 \Rightarrow t = \sqrt{\frac{100}{9.81}} = 3.19$ sec
and impact velocity $v(3.19) = -9.81(3.19) = -31.32$ m/sec.

4 (15pts) $x'' + x' + x = 2\sin(2t)$. $x(t) = A\sin 2t + B\cos 2t$. $x' = 2A\cos 2t - 2B\sin 2t$
$x'' = -4A\sin 2t - 4B\cos 2t$. $x'' + x' + x = (-4A - 2B + A)\sin 2t + (-4B + 2A + B)\cos 2t = 2\sin 2t$
$\Rightarrow -3A - 2B = 2$, $2A - 3B = 0$ Solve for $A, B \Rightarrow A = -\frac{6}{13}$, $B = -\frac{4}{13}$. $x(t) = -\frac{6}{13}\sin 2t - \frac{4}{13}\cos 2t$

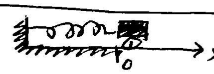
5 (15pts) 6L/min @ 2kg/L time of overflow: $2t = 200 \Rightarrow t = 100$ min.



$x(t)$ — amount of salt at time t, $x(0) = 0$ $0 \le t \le 100$

$\Rightarrow \begin{cases} \frac{dx}{dt} = 6L/\text{min}(2\text{kg}/L) - 4L/\text{m}\,\frac{x(t)}{100+2t}\,\text{kg}/L \\ x(0) = 0 \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = 12 - \frac{4x}{100+2t} \\ x(0) = 0 \end{cases}$

6 (10pts)

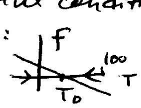


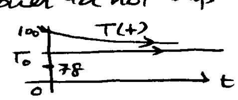
(a) $0.1\,N/\text{cm} = 10\,N/\text{m} = k$.
(b) $b = 3N$-sec/m. $m = 2kg \Rightarrow \begin{cases} mx'' + bx' + kx = 0 \\ x(0) = 0,\ x'(0) = 0.5\,\text{m/sec} \end{cases} \Rightarrow \begin{cases} 2x'' + 3x' + 10x = 0 \\ x(0) = 0,\ x'(0) = \frac{1}{2} \end{cases}$

7 (20pts) (a) $\begin{cases} \frac{dT}{dt} = k(85 - T) \\ T(0) = 78 \end{cases}$, $T(t) = Ce^{-kt} + 85 = -7e^{-kt} + 85$ w/ $T(0) = 78$. $80 = T(10)$
$\rightarrow 80 = -7e^{-k10} + 85$, $\Rightarrow k = -\frac{1}{10}\ln\frac{5}{7} = 0.034$

(b) $\begin{cases} \frac{dT}{dt} = 0.034(100 - T) - 0.5 \\ T(0) = 100 \end{cases}$

(c) Solve the equation: homogeneous part: $T_h = Ce^{-kt}$, nonhomogeneous, particular sdu: $T_p = 100 - \frac{0.5}{0.034} = 85.29 \Rightarrow T(t) = Ce^{-kt} + 85.29 = 14.71e^{-kt} + 85.29$.
which is always > 85.29 for all $t > 0$. So $T(t) \neq 78$ °F. Never. If the temperature outside stays at 100°F for a long time, Bill's air conditioner will not bring the temperature down to 78°F. He needs a more powerful conditioner for hot days of 100°F+. (alternatively using phase line analysis:
$\frac{dT}{dt} = f(T) = 0.034(100 - T) - 0.5 = 0 \Rightarrow T_0 = 85.29$.





$\Rightarrow T(t)$ never reaches 75°F.)

(End)