1(20pts) Consider the system of equations $x' = Ax$ with $A = \begin{bmatrix} -5 & -6 \\ 3 & 4 \end{bmatrix}$

(a) Find a general solution of the system.

(b) Sketch a phase portrait of the system, including all straight line solutions, and a few typical solutions.

2(15pts) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 3 \\ 2 & -3 & 1 \end{bmatrix}$$

and eigenvectors corresponding to the complex eigenvalues only.

3(15pts) It is given that $\lambda = -2 + i$ is an eigenvalue of a real valued $2 \times 2$ matrix $A$ and $u = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$ is a corresponding eigenvector. Find the solution to the initial value problem

$$x' = Ax, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

4(20pts) Sketch a phase portrait of the system of equations

$$\begin{cases} x' = y \\ y' = x - y \end{cases}$$

by including its nullclines, typical vector fields on and off the nullclines, separatrix if any, and a few typical solution curves. Must the separatrix solutions be straight lines? why or why not?

5(15pts) Consider a cooperative system of two species

$$\begin{cases} x' = x(1 - x + y) \\ y' = y(1 - y + 0.5x) \end{cases}$$

For $x(0) > 0, y(0) > 0$, will the solution converge to a co-existence state? Use a phase plane analysis to answer this question.

6(15pts) Consider a competitive system of two species

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(1 - y - 0.5x) \end{cases}$$

Is co-existence possible? Which specie must die out and under what condition? Base your conclusion on a phase plane analysis of the system.

Bonus 5pts: The state flower of Nebraska is ________________

The End
Math 221 Exam 4, Summer 2002

1. (20pts) (a) \[ A = \begin{bmatrix} \frac{3}{2} & -\frac{4}{3} \\ \frac{3}{2} & \frac{1}{3} \end{bmatrix} \] \[ \det(A - \lambda I) = \begin{vmatrix} \frac{3}{2} - \lambda & -\frac{4}{3} \\ \frac{3}{2} & \frac{1}{3} - \lambda \end{vmatrix} = (x + 1 - \lambda) (x + 1 - \lambda) = 0, \]
\[ \lambda_1 = -2, \lambda_2 = -1. \] \[ \text{Solve } (A - \lambda I) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \]
\[ \Rightarrow -u_1 + 2u_2 = 0 \Rightarrow u_1 = 2u_2 \] \[ \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]
\[ \Rightarrow \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \]
\[ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ \text{General solution: } \mathbf{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{t} \]

(b)

2. (15pts) \[ A = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} \]
\[ \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ 1 & -1 - \lambda \end{vmatrix} = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 1 \pm 3i \]
\[ \text{Solve } (A - \lambda I) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0 \]
\[ \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 - \lambda & -1 \\ 1 & -1 - \lambda \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]
\[ \Rightarrow \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

3. (15pts) \[ \mathbf{x}(t) = c_1 e^{2t} + c_2 e^{(-1+3i)t} \]
\[ \Rightarrow \mathbf{x}(t) = c_1 e^{2t} + c_2 (e^{3it} + i e^{-3it}) \]
\[ \Rightarrow \mathbf{x}(t) = c_1 e^{2t} + c_2 e^{3it} \]
\[ \Rightarrow \mathbf{x}(t) = c_1 e^{2t} + c_2 e^{3it} \]

4. (20pts) \[ x' = y, y' = -x \]
\[ x_{\text{nullcline: } y = 0, y' = 0} \]
\[ x_{\text{nullcline: } x = y, y' = 0} \]

5. (5pts) \[ \begin{cases} x' = x(1-x-y) \\ y' = y(1-x+y) \end{cases} \]
\[ x_{\text{nullcline: } x = 0, 1-x-y = 0} \]
\[ y_{\text{nullcline: } y = 0, 1-y-x = 0} \]
\[ x_{\text{species must die out as long as } y(0) > 0} \]

6. (15pts) \[ x' = x(1-x-y) \]
\[ y' = y(1-x+y-0.5x) \]

Bonus: Problem 3(3) but any answer will receive the points.