June 14, 2002

Math 221 Test 1

Summer 2002

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1(15pts) (a) Verify that \( y(x) = \sin \frac{\pi}{4}x - \ln x \) is the solution to this IVP: \( y'' + \frac{\pi}{2}y = \frac{1}{x^2} - \frac{\pi}{2} \ln x, \ y(1) = 0 \). 

(b) Does \( x^3 + xy + y^3 = 3 \) define an implicit solution to the equation \( y' + x^2y^2 = x \)?

2(15pts) Find the solution to the initial value problem \( \frac{dy}{dx} = \frac{1 + y^2}{yx^2}, \ y(1) = 1 \).

3(10pts) At what initial points \((t_0, y_0)\) does the Theorem of Existence and Uniqueness for the initial value problem: 

\[
\frac{dy}{dt} = \frac{2}{t - y}, \ y(t_0) = y_0 \text{ apply? Determine the region and verify all conditions of the theorem.}
\]

4(15pts) Consider the IVP: \( y' = \frac{1}{x}(y^2 + x), \ y(1) = -1 \).

(a) Use the Euler method to find approximate values of the solution at these points \( x = 1, 1.2, 1.4, 1.6 \).

(b) Sketch your approximating solution in the \( xy \) plane.

5(15pts) Consider the linear equation \( ty' = -2y + t^2 \).

(a) Write it in the standard form \( y' = p(t)y + q(t) \) (or \( y' + p(t)y = q(t) \)).

(b) Find a general solution to the equation.

6(15pts) (a) Consider the equation \( x' = -\sqrt{x} + t \). Sketch these isoclines: \( f(t, x) = -2, -1, 0, 1, 2 \) and the vector field on the isoclines in the upper half plane \( x > 0 \).

(b) The vector field of an equation is given below. Sketch solutions that go through these points: (i) \((-2, 0)\), (ii) \((-1, 1)\).

7(15pts) Consider the autonomous equation \( \frac{dy}{dt} = f(y) = y^3 - y \).

(a) Sketch the phase line and classify each equilibrium point as sink, source, or node,

(b) Sketch a qualitative portrait for the solutions, including the ones through \( y(0) = -1.5, -0.5, 0.5, 1.5 \).

END
1 (15 pts) (a) \( y = \sin \pi x - \ln x \), \( y' = \pi \cos \pi x - \frac{1}{x} \), \( y'' = -\pi^2 \sin \pi x + \frac{1}{x^2} \).

(b) \( y' + \frac{\pi^2}{2} y = -\frac{\pi^2}{2} \sin \pi x - \frac{\pi^2}{2} \ln x \). \( y(1) = 0 \), \( y'(1) = 0 \). 

2 (15 pts) \( \frac{dy}{dx} = \frac{1+y^2}{y} \), \( \frac{y}{y+1} \frac{dy}{dx} = \frac{1}{x} \), \( \frac{dy}{dx} \ln(x+1) = -\frac{1}{x} + C \).

3 (15 pts) \( \frac{dy}{dx} = \frac{y}{x-y} \), \( f(x,y) = \frac{y}{x-y} \), \( \frac{df}{dx} = \frac{y}{(x-y)^2} \). Both are continuous in the region \( t \pm y \), the theorem applies to any IVP with \( t \neq y \).

4 (15 pts) \( y' = \frac{1}{x} (y^2 + x) \), \( y(1) = 1 \). \( x_0 = 1 \), \( y_0 = -1 \), \( y_n = y_{n-1} + \frac{1}{x_{n-1}} (y_{n-1}^2 + x_{n-1}) \), \( h = 0.2 \).

5 (15 pts) (a) \( y' = -\frac{1}{3} y + t \). (b) \( y' + \frac{1}{3} y = t \). \( \mu(t) = e^{\frac{t^2}{3} dt} = e^{\frac{t^2}{3}} \).

6 (15 pts) (c) \( y' = -\frac{3}{5} y + t \). \( \mu(t) = e^{\frac{-3}{5} dt} = e^{\frac{-3}{5} t} \).

7 (15 pts) (c) \( \frac{dy}{dt} = y^3 + y = y(y^2 - 1) = f(y) \).