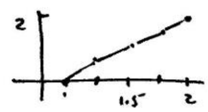
Soln. Key to Math221 Final Exam, Fall' 2015

1 (15pts) (a) $y_{k+1} = y_k + f(t_k, y_k)h$, $h = 0.25$

k	t_k	y_k	$f(t_k, y_k)h$
0	1	0	.5
1	1.25	.5	.5
2	1.5	1	.5
3	1.75	1.5	.5
4	2	2	–



2 (20pts) $\frac{dy}{dx} + \frac{1}{x+2} y = 3x$, $y(0) = 1$, $e^{\int p\,dx} = x+2$

$y(x) = Ce^{-\int p\,dx} + e^{-\int p\,dx}\int e^{\int p\,dx} g\,dx = \frac{C}{x+2} + \frac{1}{x+2}\int (x+2)3x\,dx = \frac{C}{x+2} + \frac{1}{x+2}(x^3+3x^2)$

$1 = y(0) = \frac{C}{2} + \frac{1}{2}(0)$, $C = 2$. $y(x) = \frac{2}{x+2} + \frac{x^2(x+3)}{x+2}$

3 (15pts) $2x\sin y\,dx - \cos y\,dy = 0 \Rightarrow \frac{\cos y}{2\sin y}dy = 2x$, separable. $\int \frac{\cos y}{\sin y}dy = \int 2x\,dx$

$\Rightarrow \ln|\sin y| = x^2 + C$, $\sin y = Ce^{x^2}$, $y = \sin^{-1}(Ce^{x^2})$

4 (20pts). $\begin{cases} x' = 2x - y \\ y' = 8x - y^2 \end{cases}$ (a) $\begin{cases} 2x - y = 0 \\ 8x - y^2 = 0 \end{cases} \Rightarrow y = 2x$, $8x - 4x^2 = 4x(2-x) = 0 \Rightarrow x = 0, 2 \Rightarrow$ eq. soln $(0,0)$, $(2,4)$

(b) $J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 8 & -2y \end{bmatrix}$, $J(0,0) = \begin{bmatrix} 2 & -1 \\ 8 & 0 \end{bmatrix}$, $|J - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 8 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 8 = 0$, $\lambda_{1,2} = \frac{2 \pm \sqrt{4-32}}{2} = \frac{2 \pm 2\sqrt{7}}{2} i = 1 \pm \sqrt{7} i$

$\Rightarrow$ (0,0)-spiral source, unstable. $J(2,4) = \begin{bmatrix} 2 & -1 \\ 8 & -8 \end{bmatrix}$, $|J - \lambda I| = \begin{vmatrix} 2-\lambda & -1 \\ 8 & -8-\lambda \end{vmatrix}$

$= \lambda^2 + 6\lambda - 8 = 0$. $\lambda_{1,2} = \frac{-6 \pm \sqrt{36+32}}{2} = -3 \pm \sqrt{17} = \begin{cases} -3+\sqrt{17} > 0 \\ -3-\sqrt{17} < 0 \end{cases} \Rightarrow$ (2,4) – saddle, unstable

5 (10pts) Char. eq. $2(r^3+r)(r+1)^2 = 2r(r^2+1)(r+1)^2 = 0 \Rightarrow \lambda = 0, -1, -1, i, -i$

$\Rightarrow$ general solution: $y(t) = C_1 + C_2e^{-t} + C_3te^{-t} + C_4\cos t + C_5\sin t$

6 (20pts) Solve $y'' - 2y' - 3y = 0$, $r^2 - 2r - 3 = (r+1)(r-3) = 0 \Rightarrow r_1 = -1$, $r_2 = 3$, $y_1 = e^{-x}$, $y_2 = e^{3x}$

$f_1 = 2e^x$, $y_{p_1} = x^s(Ae^x)$, $s = 0$, $f_2 = 1$, $y_{p_2} = x^s(B)$, $s = 0$. $\Rightarrow y_p(x) = Ae^x + B$

7 (20pts) Solve $y'' + y = 0$. $r^2 + 1 = 0$, $r = \pm i$, $y_1 = \cos x$, $y_2 = \sin x$.

$W[y_1, y_2] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$. $y_p = u_1(x)y_1 + u_2(x)y_2$ w/

$u_1(x) = -\int \frac{y_2 f}{aW}dx = -\int \frac{\sin x \cdot \sec^3 x}{1 \cdot 1}dx = -\int \frac{\sin x}{\cos^3 x}dx = -\frac{1}{2}\cos^{-2}x = -\frac{1}{2}\sec^2 x$.

$u_2(x) = \int \frac{y_1 f}{aW}dx = \int \cos x \cdot \sec^3 x\,dx = \int \sec^2 x\,dx = \tan x$

$\Rightarrow y_p(x) = -\frac{1}{2}\sec^2 x \cdot \cos x + \tan x \cdot \sin x$

8 (10pts). $\begin{pmatrix} 3+i \\ 2i \end{pmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + i\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x}_1(t) = e^t\cos 2t\,\vec{a} - e^t\sin 2t\,\vec{b} = e^t\cos 2t\begin{bmatrix} 3 \\ 0 \end{bmatrix} - e^t\sin 2t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{x}_2(t) = e^t\sin 2t\begin{bmatrix} 3 \\ 0 \end{bmatrix} + e^t\cos 2t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. $\vec{x}(t) = C_1e^t\begin{bmatrix} 3\cos 2t - \sin 2t \\ -2\sin 2t \end{bmatrix} + C_2e^t\begin{bmatrix} 3\sin 2t + \cos 2t \\ 2\cos 2t \end{bmatrix}$

9 (20pts) $\vec{x}' = \begin{bmatrix} 0 & -1 & 1 \\ 14 & 5 & -19 \\ 2 & 0 & -2 \end{bmatrix}x$, $|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & 1 \\ 14 & 5-\lambda & -19 \\ 2 & 0 & -2-\lambda \end{vmatrix} = -\lambda\begin{vmatrix} 5-\lambda & -19 \\ 0 & -2-\lambda \end{vmatrix} + \begin{vmatrix} 14 & -19 \\ 2 & -2-\lambda \end{vmatrix} + \begin{vmatrix} 14 & 5-\lambda \\ 2 & 0 \end{vmatrix}$

$= -\lambda^3 + 3\lambda^2 - 2\lambda = -\lambda(\lambda^2 - 3\lambda + 2) = -\lambda(\lambda-1)(\lambda-2) = 0 \Rightarrow \lambda = 0, 1, 2$.

$\lambda = 0$: $(A - \lambda I)\vec{v} = \begin{bmatrix} 0 & -1 & 1 \\ 14 & 5 & -19 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} v_2 = v_3 \\ v_1 = v_3 \end{matrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\lambda = 1$: $(A - \lambda I)\vec{v} = \begin{bmatrix} -1 & -1 & 1 \\ 14 & 4 & -19 \\ 2 & 0 & -3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} -v_1 - v_2 + v_3 = 0 \\ 14v_1 + 4v_2 - 19v_3 = 0 \\ 2v_1 - 3v_3 = 0 \end{matrix} \Rightarrow \begin{matrix} v_3 = -2v_2 \\ v_1 = -3v_2 \end{matrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\lambda = 2$: $(A - \lambda I) = \begin{pmatrix} -2 & -1 & 1 \\ 14 & 3 & -19 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Rightarrow \begin{matrix} -2v_1 - v_2 + v_3 = 0 \\ 14v_1 + 3v_2 - 19v_3 = 0 \\ 2v_1 - 4v_3 = 0 \end{matrix} \Rightarrow \begin{matrix} v_2 = -3v_3 \\ v_1 = 2v_3 \end{matrix} \Rightarrow \vec{v}_3 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

$\Rightarrow$ General solution $\vec{X}(t) = c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2 e^{t}\begin{bmatrix}3\\-1\\2\end{bmatrix} + c_3 e^{2t}\begin{bmatrix}2\\-3\\1\end{bmatrix}$

10 (15 pts) $\dfrac{21s+20}{s(s^2+6s+10)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+6s+10} = \dfrac{A(s^2+6s+10)+s(Bs+C)}{s(s^2+6s+10)} \Rightarrow 21s+20 = A(s^2+6s+10) + s(Bs+C)$

$s=0: 20 = A(10),\ A=2. \Rightarrow -2s^2+9s = Bs^2+Cs,\ B=-2,\ C=9.$

$F(s) = \dfrac{2}{s} + \dfrac{-2s+9}{(s+3)^2+1} = \dfrac{2}{s} + \dfrac{-2(s+3)+15}{(s+3)^2+1} \xrightarrow{\mathcal{L}^{-1}} 2-2e^{-3t}\cos t + 15e^{-3t}\sin t$

11 (15 pts) $\mathcal{L}\{u(t-\pi)\,te^{2t}\sin 3t\} = \mathcal{L}\{e^{2t}t\cdot u(t-\pi)\sin(3t)\} = \mathcal{L}\{t\,u(t-\pi)\sin 3t\}(s-2)$

$= \left(-\dfrac{d}{ds}\mathcal{L}\{u(t-\pi)\sin 3t\}\right)(s-2) = \left(-\dfrac{d}{ds}e^{-\pi s}\mathcal{L}\{\sin(3(t+\pi))\}\right)(s-2)$

$= \left(\dfrac{d}{ds}e^{-\pi s}\mathcal{L}\{-\sin(3t)\}(s)\right)(s-2) = \left(\dfrac{d}{ds}e^{-\pi s}\dfrac{3}{s^2+9}\right)(s-2)$

$= 3\dfrac{-\pi e^{-\pi s}(s^2+9) - e^{-\pi s}\cdot 2s}{(s^2+9)^2}\Bigg|_{s\to s-2} = \left[-3e^{-\pi s}\dfrac{\pi s^2+2s+9\pi}{(s^2+9)^2}\right]\Bigg|_{s\to s-2} = -3e^{-\pi(s-2)}\dfrac{\pi(s-2)^2+2(s-2)+9\pi}{((s-2)^2+9)^2}$

12 (20 pts) $y''+2y'+y = 2\delta(t-2),\ y(0)=0,\ y'(0)=1,\ s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2(s\mathcal{L}\{y\} - y(0))$

$+\mathcal{L}\{y\} = (s^2+2s+1)\mathcal{L}\{y\} - 1 = 2e^{-2s},\ \mathcal{L}\{y\} = \dfrac{1}{s^2+2s+1} + 2e^{-2s}\dfrac{1}{s^2+2s+1}$

$\cancel{\dfrac{1}{s^2+2s+2} = \dfrac{1}{(s+1)^2+1} \xrightarrow{\mathcal{L}^{-1}} e^{-t}\sin t \Rightarrow y(t) = e^{-t}\sin t + 2u(t-2)\mathcal{L}\{e^{-t}\sin t\}}$

$\cancel{y(t) = e^{-t}\sin t + 2u(t-2)e^{-(t-2)}\sin(t-2)}$

$\dfrac{1}{s^2+2s+1} = \dfrac{1}{(s+1)^2} \xrightarrow{\mathcal{L}^{-1}} te^{-t} \Rightarrow y(t) = te^{-t} + 2u(t-2)(t-2)e^{-(t-2)}$