Math 208 Quiz 3

Name:______________________________ Score:________________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, formula sheets allowed.

1. (*4*) If $f_x(1,0) = -2, f_y(1,0) = 1$, find the unit direction at which f increases most rapidly at $(1,0)$ and the maximal rate.

   Gradient: $\nabla f(1,0) = \langle -2, 1\rangle$. The unit directional vector of the gradient: $\vec{u} = \frac{\nabla f(1,0)}{\|\nabla f(1,0)\|} = \boxed{\langle \frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle}$. In this direction, the rate of change is maximal with the maximal rate $\|\nabla f(1,0)\| = \boxed{\sqrt{5}}$

2. (*4*) It is given that x can be solved as a function of y, z from the equation $2xe^{xy} + xz^2 + yz = 3$ at the point $(1,0,-1)$. Use implicit differentiation to find $\frac{\partial x}{\partial z}(0,-1)$ at the point.

   Let $F(x,y,z) = 2xe^{xy} + xz^2 + yz$. Then $\frac{\partial x}{\partial z}(0,-1) = -\frac{F_z(1,0,-1)}{F_x(1,0,-1)} = -\frac{2xz+y}{2e^{xy}+2xye^{xy}+z^2} = \boxed{\frac{2}{3}}$

3. (*4*) Verify that $(1,1,2)$ is on the level surface $xy + xz - yz = 1$. Find an equation of the tangent plane to the surface at the point.

   Checked: $1(1) + 1(2) - 1(2) = 1$. Let $f(x,y,z) = xy + xz - yz$. Then a normal vector to the surface and the tangent plane is $\nabla f(1,1,2) = \langle 3, -1, 0\rangle$. The equation: $3(x-1) - 1(y-1) + 0(z-2) = 0$ or $\boxed{3x - y - 2 = 0}$

4. (*4*) Find the directional derivative of $f(x,y) = xy^2$ at $(1,2)$ in the direction towards $(2,0)$.

   $f_x(1,2) = 4, f_y(1,2) = 4$. $\nabla f(1,2) = \langle 4, 4\rangle$. Vector from $(1,2)$ towards $(2,0)$: $\langle 2-1, 0-2\rangle = \langle 1, -2\rangle$. The unit directional vector: $\vec{u} = \langle \frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\rangle$. Directional derivative

   $$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \boxed{-\frac{4}{\sqrt{5}}}$$

5. (*4*) Find all critical points of $f(x,y) = x^2 + xy^2 - 2y^2 - 6x$.

   Solve

   $$\begin{cases} f_x(x,y) = 2x + y^2 - 6 = 0 \\ f_y(x,y) = 2xy - 4y = 0 \end{cases} \quad (1)$$

   by first factorizing the second equation $2y(x-2) = 0$ which leads to two branches: $y = 0$ and $x = 2$. For $y = 0$, the first equation gives $x = 3$. For $x = 2$, the first equation gives $y = \pm\sqrt{2}$. Critical points are: $\boxed{(3,0), (2,\sqrt{2}), (2,-\sqrt{2})}$

END