Math107 Fall 2011
Calculus II University of Nebraska-Lincoln

Project

The Brachistochrone Problem

The birth of the calculus of variations, a "calculus in infinite dimensions", is often associated with the following challenge issued in *Acta Eroditorum*, Volume 15 (1697):

> *I,* JOHANN BERNOULLI*, greet the most clever mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem whose solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.... Given two points P and Q in a vertical plane, what is the curve that a bead must follow such that starting from P, it reaches Q in the shortest possible time under its own gravity.*

Without loss of generality, we assume that $P = (0,0)$ and $Q = (b, B)$ with $b > 0$ and $B < 0$ (see Figure 1).

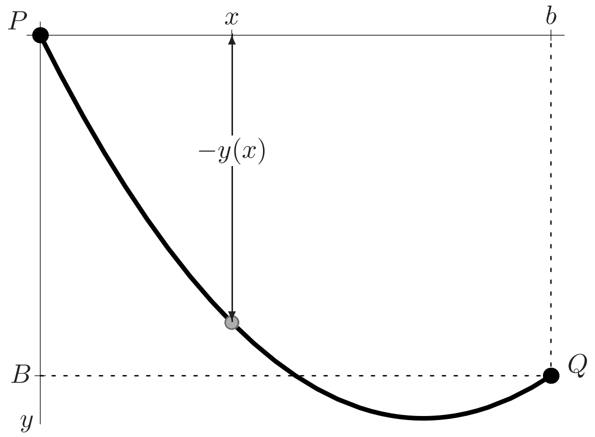

Figure 1: Setup for the brachistochrone problem

Next, we establish some notation. Denote the total time to travel from P to Q by T. Also let g be the acceleration due to gravity (which we assume to be a positive constant). Assume that the curve the bead travels along is the graph of a function y that is differentiable on the interval $(0, b)$, with $y(x) < 0$ for all $x \in (0, b]$.

With this setup, a formula for T can be found in terms of the function y. For details, see the appendix at www.math.unl.edu/˜mfoss/appendix1 or Section 9.3 in the text, but for convenience, the formula is

$$T = \sqrt{\frac{1}{2g}} \int_0^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-y(x)}} \, dx.$$

The brachistochrone problem can be stated in the following way: find a function y that minimizes the travel time T among all functions y that are differentiable on the interval $(0, b)$ and satisfy

$$y(0) = 0 \quad \text{and} \quad y(b) = B.$$

Problem Set

For the following problems, distance is measured in feet and time is measured in seconds. If you would like some additional guidance for problems 4 and 5, see www.math.unl.edu/~mfoss/hint.

1. Suppose that the bead travels from the point $(0,0)$ to the point $(1,-2)$ along the graph of a power function $y = -2x^p$, with $p > 0$. The formula for T then involves an improper integral. For which values of $p > 0$ is the integral convergent?

2. Find the exact time it takes for the bead to travel from the point $(0,0)$ to the point $(1,-2)$, if it follows a straight line.

3. (a) For each $R > 0$ and $0 < \theta_1 \leq 2\pi$, compute the exact time it takes for the bead to travel from the point $\big(0,0\big)$ to the point $\Big(R(\theta_1 - \sin\theta_1), R(\cos\theta_1 - 1)\Big)$, if it follows a cycloid: the path described by the parametric curve

$$x(\theta) = R(\theta - \sin\theta) \text{ and } y(\theta) = R(\cos\theta - 1) \quad \text{for } \theta \text{ in } [0, \theta_1].$$

   For more information about cycloids, see www.math.unl.edu/~mfoss/appendix2 or Example 3 in Section 9.6 of the text.

   (b) Estimate the value of R and θ_1 satisfying

$$\Big(R(\theta_1 - \sin\theta_1), R(\cos\theta_1 - 1)\Big) = \big(1, -2\big).$$

   Include an upper bound for the error in your estimate. (One option is to use the bisection method. You can also use a graphing calculator as described in Section 2.6.)

   (c) Using your results from (a) and (b), estimate the time it takes for the bead to travel from the point $(0,0)$ to the point $(1,-2)$, if it follows the path of a cycloid. Include an upper bound for the error in your estimate.

4. Suppose that the bead travels from $(0,0)$ to $(1,-2)$ along the graph of $y = -2\sqrt{x}$. Estimate the time it takes for the bead to make the trip. Include an upper bound for the error in your estimate.

5. If the bead travels from $(0,0)$ to $(1,-2)$ along the graph of $y = -2x^{\frac{3}{5}}$, how much time does it take? Include an upper bound for the error in your estimate.

6. Provide a careful sketch with the different paths of the bead described above. Among the curves described in Problems 2 through 5, which one provides the shortest travel time for the bead? You may need to refine your estimates so that the upper bounds on each of your errors is sufficiently small to make a comparison possible.