# Appendix 1: The Brachistochrone Problem

## Statement of the Brachistochrone Problem

The birth of the calculus of variations is often associated with the following challenge issued by JOHANN BERNOULLI in 1696: *Let two horizontally and vertically separated points $P$ and $Q$ be given in a plane with gravity acting downward. Find the curve joining the two points such that a bead starting from rest at the higher point will slide without friction along the curve and reach the lower point in the shortest possible time* (Figure 1).

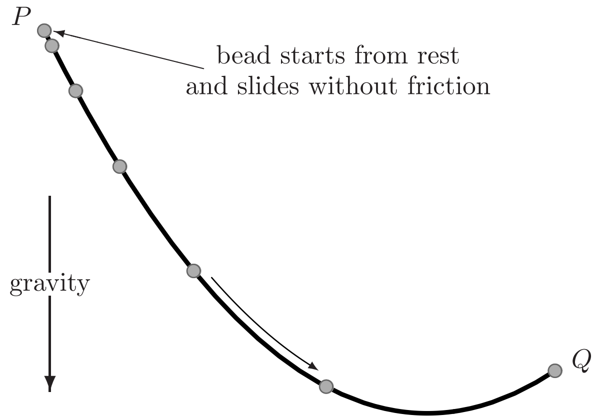


Figure 1: Illustration for the statement of the brachistochrone problem. The position of the bead is shown at equal time intervals.

## Mathematical Formulation of the Brachistochrone Problem

Without loss of generality, we assume that $P = (0,0)$ and $Q = (b,B)$ with $b > 0$ and $B < 0$ (see Figure 2).

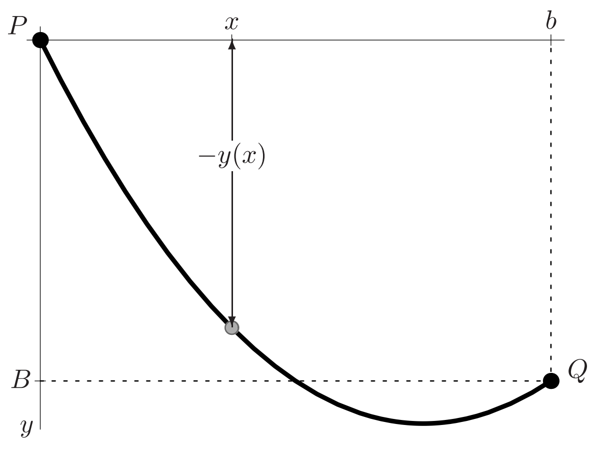


Figure 2: Setup for the brachistochrone problem

Next, we establish some notation. Denote the mass of the bead by $m$, the total time to travel from $P$ to $Q$ by $T$, and the speed of the bead by $v$. Also let $t$ be time and $g$ be the unsigned acceleration due to gravity (which we assume to be a positive constant). Assume that the curve the bead travels along is the graph of a function $y$ that is differentiable on the interval $(0, b)$, with $y(x) < 0$ for all $x \in (0, b]$.

Here is a summary of our assumptions:

(1) $P = (0, 0)$ and $Q = (b, B)$ with $b > 0$ and $B < 0$;

(2) the acceleration due to gravity is a positive constant $g$;

(3) the curve along which the bead travels is the graph of a function $y$ that is differentiable on $(0, b)$.

(4) $y(x) < 0$ for each $x \in (0, b]$.

Our goal is to find a formula for $T$ in terms of the bead's path described by $y$. To this end, we let $s$ be the total arclength of the graph of $y$ (see Sections 5.4 and 9.3 in the text), so

$$s = \int_0^b \sqrt{1 + y'(x)}\, dx.$$

We define $l$ to be the arclength function for $y$; i.e.

$$l(x) = \int_0^x \sqrt{1 + y'(\tau)^2}\, d\tau, \qquad \text{for each } x \in [0, b].$$

The value of $l(x)$ tells us the arclength of the graph of $y$ over the interval $[0, x]$. Notice that $l(0) = 0$ and $l(b) = s$. Also, by the Fundamental Theorem of Calculus $\frac{dl}{dx} = \sqrt{1 + y'(x)^2}$, so

$$dl = \sqrt{1 + y'(x)^2}\, dx. \tag{1}$$

Although we need to think of the velocity of the bead as a function of $x$, the value of $v$ is actually telling us how quickly the bead is traveling along the graph of $y$. Thus $v = \frac{dl}{dt}$, or in terms of differentials, we have $dt = \frac{1}{v} dl$. The total travel time for the bead is

$$T = \int_0^T dt.$$

We want the right-hand side to be expressed as an integral with respect to $x$. We achieve this by making some substitutions. Each time we make a change of variables, we must also make an appropriate change in the limits of integration (recall Section 4.6). Here we go. First we change from integration in time to integration along the graph of $y$: the total time of travel for the bead is

$$T = \int_0^T dt = \int_0^s \frac{1}{v}\, dl,$$

since at time $T$ the bead is at the end of the graph of $y$ and has traveled a distance $s$; i.e. $l = s$ at time $T$. Now, we change from integration along the graph of $y$ to integration in $x$: using (1) gives us

$$T = \int_0^s \frac{1}{v}\, dl = \int_0^b \frac{\sqrt{1 + y'(x)^2}}{v(x)}\, dx, \tag{2}$$

since when $l = s$, we are at the end of the graph of $y$ and so $x = b$. To express $T$ completely in terms of the function $y$, our ultimate goal, we need to find a relation between $v$ and $y$. We use the fact that the total energy of the bead is conserved: (Kinetic Energy)+(Potential Energy) is constant. Kinetic energy is given by $\frac{1}{2}mv^2$, while the potential energy of the bead is $mgy$. Thus the conservation of energy implies

$$\frac{1}{2}mv(x)^2 + mgy(x) = C \qquad \text{for each } x \in [a, b],$$

where $C$ is some constant. Since the bead starts from rest at the point $P = (0, 0)$, we deduce that $y(0) = 0$ and $v(0) = 0$. Thus $C = 0$, and

$$\frac{1}{2}mv(x)^2 + mgy(x) = 0 \Rightarrow v(x)^2 = -2gy(x) \Rightarrow v(x) = \sqrt{-2gy(x)} \quad \text{for each } x \in [a, b]. \tag{3}$$

Note that we have assumed that $v \geq 0$. Plugging the result in (3) into the formula (2) yields

$$T = \sqrt{\frac{1}{2g}} \int_0^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-y(x)}}\, dx. \tag{4}$$

This is our expression for the travel time of the bead in terms of the function $y$.

Now we can restate the brachistochrone problem in a more useful way. Find the function that minimizes the value of

$$T = \sqrt{\frac{1}{2g}} \int_0^b \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-y(x)}}\, dx$$

among all functions $y$ that are differentiable on the interval $(0, b)$ and satisfy

$$y(0) = 0 \quad \text{and} \quad y(b) = B.$$

It is actually sufficient for $y$ to be differentiable on $(0, b)$ except at a finite number of points, so, for example, $y$ can have a finite number of corners.