Project

There is a long history of attempts to compute and estimate the value of π. Sometime around 2000 B.C., the Egyptians obtained the earliest known approximation of $\left(\frac{4}{3}\right)^4$. During the 1600's, James Gregory produced the following formula for the value of $\frac{\pi}{4}$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

This can be seen to be the Taylor series, centered at 0, for $\tan^{-1}(x)$ evaluated at 1. In 1706, John Machin provided an alternative formula based on

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right).$$

Using this formula, John Machin accurately computed the first 100 digits of π by hand. Later other formulae for computing π were found. Here is one used by Karl Gauss:

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right).$$

For this project, you will develop a method for computing the number π and compare it to Gregory's formula.

Problem Set

1. Compute $P_2(x)$: the Taylor polynomial of degree 2, centered at 0, for $\tan^{-1}(x)$. Given $r > 0$, estimate the maximum error between $P_2(x)$ and $\tan^{-1}(x)$ over the interval $[-r, r]$.

2. Estimate how many terms from Gregory's series will be needed to approximate π accurate to one hundred decimal places. Estimate how many terms are needed to ensure an approximation accurate to one thousand decimal places. What about ten thousand?

3. Suppose that x and y are numbers such that

$$\frac{\pi}{4} = \tan^{-1}(x) + \tan^{-1}(y). \qquad (*)$$

   Provide a formula for $\frac{\pi}{4}$ as a sum of two series: one involving x and the other involving y. Explain why $\frac{\pi}{4}$ can be better approximated, using fewer terms from your two series, if x and y are as small in magnitude as possible.

4. Using the identity

$$\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)},$$

   solve (*) for y in terms of x. Find the values of x and y that satisfy (*) and such that the point (x, y) is as close to the origin as possible.

5. Select rational numbers satisfying (*) that are close to the the values of x and y you found in problem 4. Using your result from problem 3, provide a formula for $\frac{\pi}{4}$ as a sum of two infinite series.

6. Estimate how many terms from this new series will be needed to approximate π accurate to one hundred decimal places. Estimate how many terms are needed to ensure an approximation accurate to one thousand decimal places. What about ten thousand?

7. Suppose that you have written a computer program to compute partial sums for Gregory's series as well as the new series formula that you produced in problem 5. If the program requires 1 microsecond to compute a new term and add it to the previously computed terms, how long will it take to use Gregory's formula to accurately compute π to ten thousand decimal places? How long will it take using your new formula? Which of the series is the more effective for approximating π.