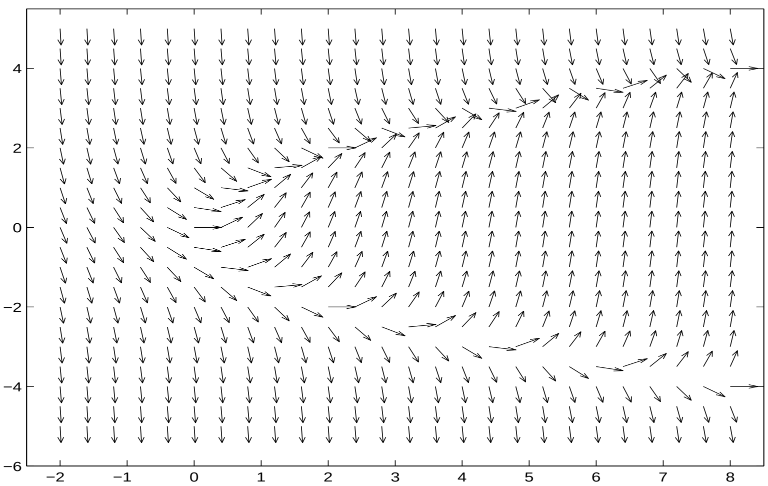
Dec. 5, 2002	MATH 107 Sec 251–257 Exam III	Fall Semester, 2002

Name:________________ Section:__________ Score:__________

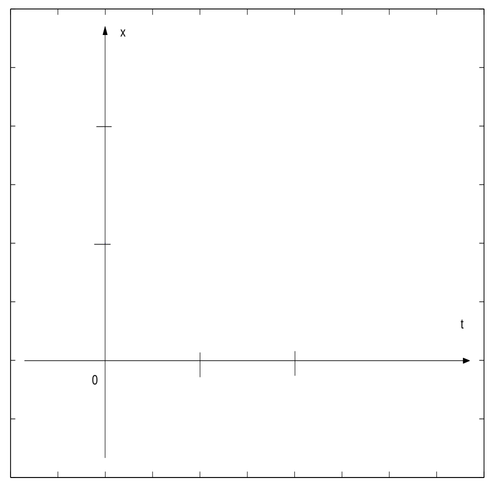
Instructions: You must show supporting work to receive full and partial credits.

1(10pts) (a) The vector field of an equation is given below. Sketch solutions that go through these points: $(-2, 0)$, $(-1, 1)$.

(b) On the field provided below plot the vector field of the equation $\frac{dx}{dt} = t - x$ at these two points $(t, x) = (1, 1), (2, 1)$.



(a)



(b)

2(10pts) (a) Find all equilibrium solutions to the equation $\dfrac{dP}{dt} = 10P - 5P^2$.

(b) Verify whether or not $y(x) = x^2 + \sin x$ is a solution to the equation $\frac{d^2y}{dx^2} = -y + 2$

(*Continue on Next Page ...*)

3(15pts) Use Euler's method to approximate the solution to the initial value problem $\begin{cases} \dfrac{dx}{dt} = t - x \\ x(1) = 2 \end{cases}$ at $x(2)$, using step size $\Delta t = 0.5$. **Show all steps and calculator programs are NOT allowed.**

4(15pts) (a) Find the general solution to the equation $\frac{dz}{dt} = z + zt^2$ by the method of separation of variables.

(b) Find the solution that satisfies the initial condition $z(0) = 5$.

(*Continue on Next Page ...*)

5(15pts) Water leaks from a vertical cylindrical tank through a small hole in its base at a rate proportional to the square root of the volume of water remaining. If the tank initially contains 100 liters and 19 liters leak out during the first day, when will the tank be half empty?

6(17pts) Given the points $P = (0, 1, 0), Q = (-1, 1, 2), R = (2, 1, -1)$.

(a) Find the area of the triangle ΔPQR.

(b) Find the equation of a plane that contains $P,\ Q,\ R$.

(*Continue on Next Page ...*)

7(18pts) Let $\vec{u} = (1, 2, 0), \vec{v} = (3, -1, 2)$.

(a) Find the unit vector that is opposite to the direction of $\vec{u}$.

(b) Find the cosine of the angle between $\vec{u}$ and $\vec{v}$.

(c) Find the equation of a plane that is parallel to the plane $2x + 4y - 3z = 1$ and through the point $(1, 0, -1)$.

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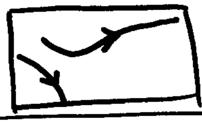
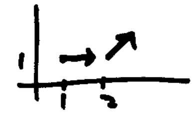
2 Bonus Points: Find the project of $\vec{u}$ on $\vec{v}$ for $\vec{u}, \vec{v}$ given in #7 above.

The End

Solution Key to Test II Math 107 Spring '02

1 (10 pts) (a) 

(b) $f(t,x) = t - x$, $f(1,1) = 0$, $f(2,1) = 1$ 

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2 (10 pts) (a) $f(p) = 10p - 5p^2 = 5p(2-p) = 0 \Rightarrow$ $p = 0$ and $p = 2$

(b) leftside $= \frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(x^2 + \sin x) = \frac{d}{dx}(2x + \cos x) = 2 - \sin x$. Right side $= -y + 2 = -(x^2 + \sin x) = -\sin x - x^2 \neq 2 - \sin x$. $y = x^2 + \sin x$ not a solution

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3 (15 pts) $\frac{dx}{dt} = t - x = f(t,x)$. $\Delta t = 0.5$, $t_0 = 1$, $x_0 = 2$.

n	t_n	x_n	$\Delta t\, f(t_n, x_n) = 0.5(t_n - x_n)$
0	1	2	$0.5(1-2) = -0.5$
1	1.5	$2 - 0.5 = 1.5$	$0.5(1.5 - 1.5) = 0$
2	2	1.5	—

$x(2) \approx 1.5$

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4 (15 pts) (a) $\frac{dz}{dt} = z + zt^2 = z(1+t^2)$. $\frac{dz}{z} = (1+t^2)dt$, $\int \frac{dz}{z} = \int (1+t^2)dt$

$\ln|z| = t + \frac{1}{3}t^3 + c_1$ $|z| = e^{t + \frac{1}{3}t^3 + c} = e^{c_1} e^{t + \frac{1}{3}t^3} = c_2 e^{t + \frac{1}{3}t^3}$

$\Rightarrow$ $z = ce^{t + \frac{1}{3}t^3}$ (b) $5 = z(0) = ce^{0 + \frac{1}{3}0^3} = c$. $\Rightarrow$ $z = 5e^{t + \frac{1}{3}t^3}$

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5 (15 pts). Let $V(t)$ be the water volumn. Then $V(0) = 100$ and

$\frac{dV(t)}{dt} = k\sqrt{V(t)}$. $\frac{dV}{\sqrt{V}} = k\,dt \Rightarrow 2\sqrt{V} = kt + c$. $2\sqrt{V(0)} = k(0) + c$

$\Rightarrow 2\sqrt{100} = c \Rightarrow c = 20$. At $t = 1$, $2\sqrt{V(1)} = k \cdot 1 + 20$, $k = 2\sqrt{V(1)} - 20$

$= 2\sqrt{81} - 20 = 2(9) - 20 = -2$. $\Rightarrow 2\sqrt{V} = -2t + 20$, $V(t) = (-t+10)^2$

Solve $V(t) = (-t+10)^2 = 50 \Rightarrow -t + 10 = \sqrt{50}$, $t = 10 - \sqrt{50} =$ 2.9 ~ 3 days.

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6 (17 pts) (a) $\vec{PQ} = (-1, 0, 2)$, $\vec{PR} = (2, 0, -1)$. $\vec{w} = \vec{PQ} \times \vec{PR}$

$= \begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ 2 & 0 & -1 \end{vmatrix} = 0i - (1-4)j + 0k = 3j$. area of $\Delta PQR = \frac{1}{2}\|\vec{w}\| = \frac{3}{2}$

(b) $\vec{w} \cdot (x, y-1, z) = 3(y-1) = 0 \Rightarrow$ $y = 1$

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7 (18 pts) (a) $\vec{w} = -\frac{\vec{u}}{\|\vec{u}\|} = -\frac{(1,2,0)}{\sqrt{1^2+2^2}} = \left(-\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, 0\right)$. ~~(b) cos θ~~

(b) $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1(3) + 2(-1) + 0(2)}{\sqrt{5}\sqrt{14}} = \frac{1}{\sqrt{70}}$

(c) $\vec{n} = (2, 4, -3)$ normal vector, $\vec{n} \cdot (x-1, y-0, z-(-1)) =$

$= 2(x-1) + 4y - 3(z+1) = 2x + 4y - 3z - 5 = 0$. $\Rightarrow$ $2x + 4y - 3z = 5$

Bonus $\vec{u}_{parallel} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{1}{14}(3, -1, 2)$