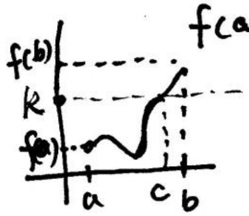
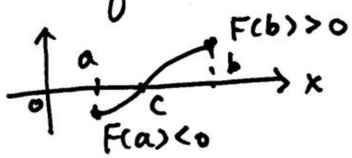
3.10 Differentiable Function.

Recall: <u>Intermediate</u> <u>Value</u> Theorem:

If f is a continuous in $[a,b]$, then for any value k between $f(a)$ and $f(b)$, then there must be a point $a \le c \le b$ so that $f(c) = k$



Appl. It is used to determine solutions to equations. $F(x) = 0$



Mean Value Theorem: If f is a continuous function in $[a, b]$ and is differentiable in (a, b), then there must be a point $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

i.e. $f(b) - f(a) = f'(c)(b-a)$.

<u>Corollary 1</u>:

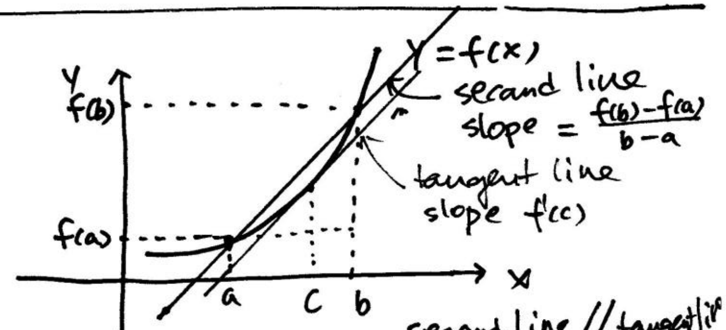
- If $f' > 0$ in $[a,b]$ at every point then f must be increasing, i.e, $f \nearrow$
- $f' < 0$ in $[a,b] \Rightarrow f \searrow$

<u>Pf</u>: Take any two points from the interval, say, $a < b$, by MVT, there is a $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

i.e. $f(b) - f(a) = \underbrace{f'(c)}_{>0}\underbrace{(b-a)}_{>0} > 0$

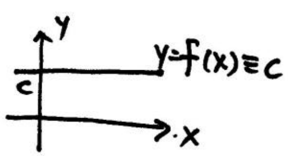
$\Rightarrow f(b) > f(a)$ i.e $f \nearrow$



secand line // tangent line

$\Leftrightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$

Corollary 2: If $f' \equiv 0$ in an interval $[a,b]$, then f must be a constant function.



Pf: Take any two points from the interval $[a,b]$, say, $a<b$. Then by MVT, there is a c w/ $a<c<b$ such that

$$f(b)-f(a) = \underbrace{f'(c)}_{0}(b-a) = 0$$

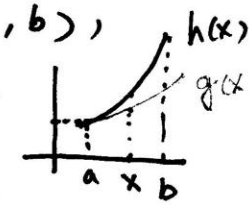
$\Rightarrow f(b) = f(a)$

$\Rightarrow f(x) = f(a)$ for all $a \le x \le b$. $\square$

The Racetrack Principle

Suppose that g and h are continuous in $[a,b]$ and differentiable in (a,b),

- If $g(a) = h(a)$ and $g'(x) \le h'(x)$, then $g(x) \le h(x)$ for $a \le x \le b$.



- If $g(b) = h(b)$ and $g'(x) \le h'(x)$, then $g(x) \ge h(x)$.

Pf: let $f(x) = h(x) - g(x)$.

Then $f'(x) = h'(x) - g'(x) \ge 0$ since $h'(x) > g'(x)$

$\Rightarrow f(x) \nearrow$

$\Rightarrow f(x) \ge f(a)$ for $x \ge a$.

$h(x) - g(x) \ge h(a) - g(a) = 0$

$\Rightarrow h(x) \ge g(x)$ $\square$.

Ex: show $\sin x \le x$ for $x \ge 0$.

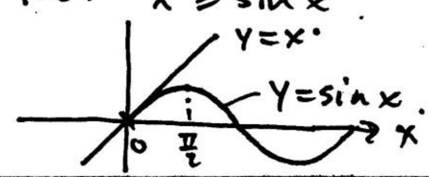
Pf: let $g(x) = \sin x$, $h(x) = x$.

$g(0) = 0 = h(0)$, and

$g'(x) = \cos x \le 1 = h'(x)$

for all x. By Racetrack Theorem, $h(x) \ge g(x)$

i.e. $x \ge \sin x$.



ID ~~(last 4 digits on your NCard)~~

Ex: Show $e^x \geq x+1$ for all x.

Pf: Consider $x \geq 0$. Let $h(x) = e^x$, $g(x) = x+1$. Then $h(0) = e^0 = 1 = g(0)$. $h'(x) = e^x \geq 1 = g'(x)$ for $x \geq 0$. By RT, $h(x) \geq g(x)$

$\Rightarrow e^x \geq x+1$ $\square$