Uniqueness of Center Manifold Dynamics

Let \( \tilde{q} \) be a nonhyperbolic fixed point of a diffeomorphism \( f \) in \( \mathbb{R}^d \). Let \( J = Df(\tilde{q}) \), and denote

\[
\sigma^s = \sigma(J) \cap \{ |z| < 1 \}, \sigma^c = \sigma(J) \cap \{ |z| = 1 \}, \text{ and } \sigma^u = \sigma(J) \cap \{ |z| > 1 \}
\]

the set of stable eigenvalues, center eigenvalues, unstable eigenvalues, respectively, of the linearization \( Df(\tilde{q}) \). Let

\[
\sigma^{cs} = \sigma^s \cup \sigma^c, \text{ and } \sigma^{cu} = \sigma^c \cup \sigma^u.
\]

Let \( \mathbb{R}^d \cong \mathbb{E}^s \times \mathbb{E}^c \times \mathbb{E}^u \) with \( \mathbb{E}^{su} \cong \mathbb{E}^s \times \mathbb{E}^u \) based at the fixed point. For \( r > 0 \), denote by \( \mathbb{E}^c_r = \{ \{ \tilde{q} \} \oplus \mathbb{E}^c \} \cap \{ \| p - \tilde{q} \| < r \} \) the \( r \)-neighborhood of \( \tilde{q} \) on its center eigenspace and similarly, \( \mathbb{E}^{su}_r = \{ \{ \tilde{q} \} \oplus \mathbb{E}^{su} \} \cap \{ \| p - \tilde{q} \| < r \} \).

**Definition 1.** A set \( W_{loc}^c \) in \( N_r(\tilde{q}) \) is called a local center-manifold of \( \tilde{q} \) if (a) it is invariant under \( f \): \( f(W_{loc}^c) \cap N_r(\tilde{q}) \subset W_{loc}^c \), (b) it is the graph of a \( C^1 \) function \( \phi_{su} : \mathbb{E}^c_r \to \mathbb{E}^{su}_r \), and (c) \( W_{loc}^c \) is tangent to \( \mathbb{E}^c \) at \( \tilde{q} \)

\[
\mathbb{T}_\tilde{q} W_{loc}^c \cong \mathbb{E}^c,
\]

i.e., \( D\phi_{su}(\tilde{q}) \cong 0 \).

**Theorem 1** (Uniqueness of Center Manifold Dynamics). Let \( \tilde{q} \) be a nonhyperbolic fixed point of a \( C^{1,1} \) diffeomorphism \( f \) in \( \mathbb{R}^d \). Let \( W_{loc,1}^c, W_{loc,2}^c \) be two \( C^{1,1} \) local center manifolds of \( \tilde{q} \). Then there is an open neighborhood \( V \) of \( \tilde{q} \) and a \( C^1 \) invertible map \( \kappa : W_{loc,1}^c \cap V \to W_{loc,2}^c \cap V \) so that

\[
f \circ \kappa(p) = \kappa \circ f(p)
\]

for all \( p \in W_{loc,1}^c \cap V \) so long as \( f(p) \in W_{loc,1}^c \cap V \). Furthermore, if \( f \) is a \( C^{k,1} \), \( k \geq 1 \) diffeomorphism and \( W_{loc,1}^c, W_{loc,2}^c \) are \( C^{k,1} \) manifolds, then the conjugacy \( \kappa \) is of \( C^k \).

**Lemma 1.** Let \( W_{loc}^c \) be a local center manifold of a fixed point \( \tilde{q} \) of a \( C^{k,1} \), \( k \geq 1 \) diffeomorphism \( f \) in \( N_{r_0}(\tilde{q}) \). If \( W_{loc}^c = \{ \tilde{q} \} \oplus \mathbb{E}^c_{r_0} \), then for sufficiently small \( 0 < r < r_0/2 \), there is a \( C^{k,1} \) diffeomorphism \( \tilde{f} \) in \( \mathbb{R}^d \) such that the following properties hold: (i) \( \tilde{f}|_U = f \) with \( U = N_r(\tilde{q}) \); (ii) \( \tilde{f}|_{\{ \| p - \tilde{q} \| \geq 2r \}} = Df(\tilde{q}) \); (iii) the whole center eigenspace \( \{ \tilde{q} \} \oplus \mathbb{E}^c \) is invariant under \( f \); and (iv) \( \| \tilde{f} - D\tilde{f}(\tilde{q}) \|_1 \to 0 \) as \( r \to 0 \).

**Proof.** Without loss of generality we assume the fixed \( \tilde{q} \) is translated to the origin and identify \( \mathbb{R}^d \cong \mathbb{E}^c \times \mathbb{E}^{su} \) with \( \mathbb{R}^d = \mathbb{E}^c \times \mathbb{E}^{su} \). Let \( x = (x_c, x_{su}) \in \mathbb{R}^d = \mathbb{E}^c \times \mathbb{E}^{su} \) be the coordinate system for the eigenspaces splitting for which \( J := Df(\tilde{q}) = \text{diag}(A_c, A_{su}) \), \( f = (f_c, f_{su}) \), \( f_c(x) = A_c x_c + h_c(x) \) where \( h(x) = f(x) - Jx \), and \( \| x \| = \| x_c \| + \| x_{su} \| \). Because \( W_{loc}^c = \mathbb{E}^c_{r_0} \), we have \( x \in W_{loc}^c \).
iff $x_{su} = 0$ and $\|x_c\| < r_0$. Because $W_{loc}^c$ is invariant, $f(W_{loc}^c) \cap N_{r_0} = W_{loc}^c$, we have $f_{su}(x_c, 0) = 0$ for $\|x_c\| < r_0$. That is,

$$0 = f_{su}(x_c, 0) = A_{su}0 + h_{su}(x_c, 0) = h_{su}(x_c, 0).$$

Conversely, if $h_{su}(x_c, 0) = 0$ for all $x_c \in V \subset \mathbb{E}^c$ for an open set $V$ containing 0, then $V$ must be invariant for $f$, and is contained in a center-manifold of $f$.

Let $\rho_r$ be a $C^\infty$ cut-off function with $\rho_r(x) = 1$ if $\|x\| \leq r$ and $\rho_r(x) = 0$ if $\|x\| > 2r$. Define $\tilde{f} = (\tilde{f}_c, \tilde{f}_{su})$ as

$$\tilde{f}(x) = Jx + \tilde{h}(x), \quad \text{with } \tilde{h}(x) = \rho_r(x)h(x).$$

Then for $0 < r < r_0/2$, we first have $\tilde{f}|_U = f$ for $U = N_r$, i.e. $\tilde{f}$ is a global extension of $f$ on $U$ and $\tilde{f}_{\|p-q\|>2r} = J$.

Next for $x = (x_c, 0) \in W_{loc}^c$, $\|x_c\| < 2r < r_0$ we have

$$\tilde{f}_{su}(x_c, 0) = A_{su}0 + \rho_r(x_c, 0)h_{su}(x_c, 0) = \rho_r(x_c, 0) \cdot 0 = 0.$$

In addition, for $x = (x_c, 0)$ and $\|x_c\| \geq 2r$, we have

$$\tilde{f}_{su}(x_c, 0) = A_{su}0 + \rho_r(x_c, 0)h_{su}(x_c, 0) = 0 \cdot h_{su}(x_c, 0) = 0.$$

This implies the whole center eigenspace $\mathbb{E}^c$ is invariant for $\tilde{f}$.

Last, because $\tilde{f}|_U = f$, we have $D\tilde{f}(0) = Df(0) = J$ and $\tilde{f}(x) - D\tilde{f}(0)x = \rho_r(x)\tilde{f}(x) - Df(0)x$, implying $\|\tilde{f} - D\tilde{f}(\tilde{q})\|_1 \to 0$ if $r \to 0$, which also implies $\tilde{f}$ is globally invertible for small $r$. 

\textbf{Lemma 2.} Let $W_{loc}^c$ be a $C^{k,1}$, $k \geq 1$ local center manifold of a fixed point $\tilde{q}$ of a $C^{k,1}$ diffeomorphism $f$ in $N_{r_0}(\tilde{q})$. Then for sufficiently small $0 < r < r_0/2$, there is a $C^{k,1}$ local center-stable manifold $W_{loc}^{cs}$, and a $C^{k,1}$ local center-unstable manifold $W_{loc}^{cu}$ in $N_r(\tilde{q})$ so that $W_{loc}^{cs} \cap N_r(\tilde{q}) = W_{loc}^{cs} \cap N_{r_0}(\tilde{q})$. Moreover, $W_{loc}^{cs}$ is equipped with a $C^k$ stable foliation and $W_{loc}^{cu}$ is equipped with a $C^k$ unstable foliation.

\textbf{Proof.} Use the same coordinate system setup as in the proof of Lemma 1 above. Let $\rho_r$ be the same type of cut-off function as well. Let $x_{su} = \phi_{su}(x_c), x_c \in \mathbb{E}^c_{r_0}$ be the $C^{k,1}$ function for $W_{loc}^c$. Define a change of variables in $\mathbb{R}^d$, $y = g(x)$ as below

$$\begin{cases}
y_c = x_c \\
x_{su} = x_{su} - \rho_{r_0}(x_c)\phi_{su}(x_c),
\end{cases}$$

whose inverse, $x = g^{-1}(y)$ is explicitly

$$\begin{cases}
y_c = x_c \\
x_{su} = y_{su} + \rho_{r_0}(y_c)\phi_{su}(y_c).
\end{cases}$$

Because $\phi_{su}$ is $C^{k,1}$, so is $g$ and $g^{-1}$. Let $\tilde{f}(y) = g \circ f \circ g^{-1}$, which is $C^{k,1}$ as well. Then, $g$ transforms $f$’s local center manifold $W_{loc}^c = \text{graph}(\phi_{su})$ to the
flat local center manifold $\mathbb{E}_r^c = \{y_{st} = 0\} \cap N_{r_0}$ for $\tilde{f}$. By Lemma 1, let $\tilde{f}$ be the extension of $\tilde{f}$. Since $\|f - D\tilde{f}(\bar{q})\|_1 \to 0$ as $r \to 0$, the Center-Stable Manifold Theorem, the Center-Unstable Manifold Theorem, the Stable-Foliation Theorem, and the Unstable-Foliation Theorem all apply. Let $W_{cs}^{c}, W_{cu}^{c}$ denote the center-stable manifold, the center-unstable manifold, respectively. Because $\mathbb{E}_r^c$ is invariant for $\tilde{f}$ whose restricted dynamics on it and outside the unbounded region $\{\|p - \bar{q}\| \geq 2r\}$ is the linear map $A_c$, $\tilde{f}$ cannot grow in either directions of iteration faster than any geometric rate. Therefore, by the definition and uniqueness of both $W_{cs}^{c}$ and $W_{cu}^{c}$ for $\tilde{f}$, $\mathbb{E}_r^c$ must be contained by both manifolds. Transform back these manifolds by $g^{-1}$ and restrict the map $\tilde{f} = g^{-1} \circ \tilde{f} \circ g$ in a small neighborhood $N_c(\bar{q})$ to recover the required structures for $f = f|_U$ with $U = N_r(\bar{q})$. In particular, $W_{loc}^{cs}, W_{loc}^{cu}$ are $C^{k,1}$, both containing $W_{loc}^{c}$, and their foliations, $\mathcal{F}^s, \mathcal{F}^u$ are $C^k$. This completes the proof.

**Proof of Theorem 1.** We prove first a special case for which $W_{loc,1}^{c}, W_{loc,2}^{c}$ both lie on one local center-stable manifold $W_{loc}^{cs}$ equipped with a $C^k$ stable foliation $\mathcal{F}^s$, or on one local center-unstable manifold $W_{loc}^{cu}$ equipped with a $C^k$ unstable foliation $\mathcal{F}^u$. Since the proof for the latter is the same as for the former, differing only by considering the inverse of $f$, we only consider the $W_{cs}^{loc}$ case.

Because both $W_{loc,1}^{c}$ and $W_{loc,2}^{c}$ are tangent to $\mathbb{E}_r^c$ at $\bar{q}$, and the stable foliation $\mathcal{F}^s(p)$ intersects $W_{loc,1}^{c}$ and $W_{loc,2}^{c}$ transversely. The conjugacy $\kappa$ is defined as follows. For any point $p \in W_{loc,1}^{c}$, the foliation $\mathcal{F}^s(p)$ through $p$ has a unique intersection with $q \in W_{loc,2}^{c}$, denote it by $q = \kappa(p)$. Because the foliation is $C^k$ and the intersection is transversal, $\kappa$ is $C^k$ as well. Moreover, it can be seen easily that $\kappa$ is invertible because $\mathcal{F}^s$ defines an equivalence relation on $W_{loc}^{cs}$ and its intersections with both local center manifolds are unique and transversal. Furthermore, $\kappa$ commutes with $f$ because whenever it is defined, wherever $\mathcal{F}^s(f(p))$ or $\mathcal{F}^u(f^{-1}(p))$ intersects $W_{loc,2}^{c}$ is wherever $f(\mathcal{F}^s(p))$ or $f^{-1}(\mathcal{F}^u(p))$ intersects $W_{loc,2}^{c}$ by the invariance of $\mathcal{F}^s$ under $f$, showing $\kappa \circ f(p) = f \circ \kappa(p)$. This proves the special case.

Next, let $W_{loc,1}^{c}, W_{loc,2}^{c}$ be any two $C^{k,1}$ local center manifolds of $\bar{q}$. Apply Lemma 2 to obtain a $C^{k,1}$ local center-stable manifold $W_{loc,1}^{cs}$ containing $W_{loc,1}^{c}$, together with a $C^k$ stable foliation for $W_{loc,1}^{cs}$. Similarly, apply Lemma 2 to obtain a $C^{k,1}$ local center-unstable manifold $W_{loc,2}^{cu}$ containing $W_{loc,2}^{c}$, together with a $C^k$ unstable foliation for $W_{loc,2}^{cu}$. Let

$$W_{loc,3}^{c} = W_{loc,1}^{cs} \cap W_{loc,2}^{cu} \cap N_r(\bar{q}).$$

Then by the Local Center Manifold Theorem, it is a $C^{k,1}$ local center manifold. Since the dynamics of $f$ on $W_{loc,1}^{c}$ is $C^k$ conjugate to $W_{loc,3}^{c}$ for both being on $W_{loc,1}^{cs}$ by the special case, and similarly, $f$ on $W_{loc,2}^{cu}$ is $C^k$ conjugate to $W_{loc,3}^{c}$ for both being on $W_{loc,2}^{cu}$, $f$ on $W_{loc,1}^{c}$ is hence $C^k$ conjugate to $W_{loc,3}^{c}$ by conjugacy’s transitivity. This proves the theorem. \[\square\]