SINGULAR PERTURBATION OF N-FRONT TRAVELLING WAVES IN THE FITZHUGH-NAGUMO EQUATIONS

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Abstract. In this paper we consider travelling wave solutions of the FitzHugh-Nagumo equations

$$v_t = v_{xx} + f(v) - w,\ w_t = \epsilon(v - \gamma w),$$

which are singularly perturbed by the parameter $0 < \epsilon \ll 1$. The equations possess n-front travelling waves that correspond to n-front heteroclinic orbits in a travelling coordinate frame. These solutions bifurcate from a double twisted heteroclinic loop at $\epsilon = 0$. Previous works showed that the n-front bifurcation branches in the γc-parameter space, where c is the propagation speed of the travelling waves, shrink to the bifurcation loop point as $\epsilon \to 0^+$. A proof is sketched to show that these n-front bifurcation branches can be uniformly continued in a γc-parameter region independent of $\epsilon \to 0^+$.

Keywords: FitzHugh-Nagumo equations, n-front travelling waves, geometric theory of singular perturbations.

1. Introduction

The FitzHugh-Nagumo equations were introduced as a simpler model to mimic some key behaviours of Hodgkin-Huxley's equations for electrical impulses propagating along the nerve axon of the giant squid ([13, 11, 21]). They have motivated the development of numerous techniques in the studies of wave propagation, bifurcation theory, stability theory, geometric theory for singular perturbations, and in many other theories and applications, see e.g. [9, 18, 4, 10, 22, 12, 14, 7, 8, 17, 16, 19, 23, 2, 1]. The purpose of this paper is to illustrate a technique for analyzing bifurcations of multiple pulse heteroclinic orbits from a double twisted heteroclinic loop in singularly perturbed equations with application to the FitzHugh-Nagumo equations.

To motivate the problem we begin with the FitzHuge-Nagumo equations

$$v_t = v_{xx} + f(v) - w,\ w_t = \epsilon(v - \gamma w), \tag{1.1}$$

where $f(v) = v(1-v)(a-v)$ is the usual cubic function used in most of the literature. Travelling waves are solutions of the form

$$(v, w)(x, t) = (v, w)(x + ct),$$

where $c > 0$ is the wave propagation speed. Casting the equations in the travelling coordinate frame $z = x + ct$ transforms the reaction-diffusion equations Eq. (1.1) into a system of ordinary differential equations as follows

$$\begin{aligned} c\dot{v} &= \ddot{v} + f(v) - w \\ c\dot{w} &= \epsilon(v - \gamma w), \end{aligned} \tag{1.2}$$

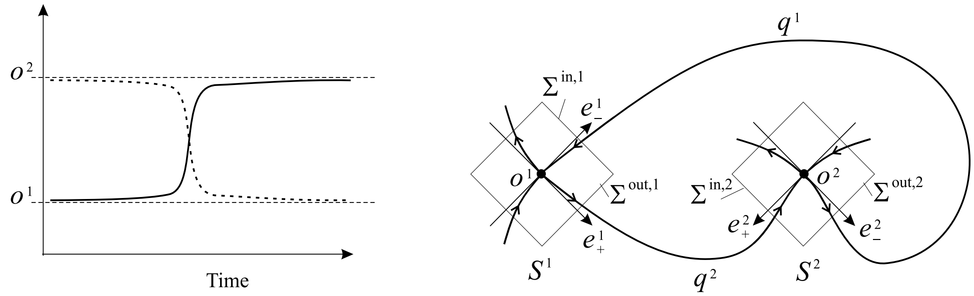

FIGURE 1. A twisted heteroclinic loop in 2-dimensional phase space (right). Typical graph in one phase variable for a heteroclinic orbit from o^1 to o^2 (solid left) and from o^2 to o^1 (dotted left).

where $\dot{} = \frac{d}{dz}$. Depending on the size of γ, there are one, two, or three equilibrium solutions of Eq. (1.2). We are interested in the case of three equilibria. Let o^1 and o^2 be two of the equilibria that will be specified in more details later. Our problem concerns with a class of travelling waves of Eq. (1.1) that correspond to heteroclinic or homoclinic solutions, $q(z)$, of Eq. (1.2) satisfying

$$\lim_{z\to\infty} q(z) = o^i \text{ and } \lim_{z\to-\infty} q(z) = o^j.$$

If $i \neq j$, $q(z)$ is a heteroclinic solution of Eq. (1.2). Otherwise it is a homoclinic solution with $i = j$.

If it travels "straight" from o^i to o^j, it is called a simple front. If at some parameter value there exist two simple fronts, one from o^1 to o^2 and the other from o^2 to o^1, then we have what we call a heteroclinic loop, see Fig. 1. A pulse homoclinic orbit of o^1 around o^2 is a homoclinic orbit of o^1 that passes through a neighborhood of o^2 in a finite z interval before converging to o^1 as $z \to \infty$, see Fig. 2. A similar definition applies to pulse homoclinic orbits of o^2 around o^1. A 1-front heteroclinic orbit from o^1 to o^2 is a heteroclinic orbit that passes through a neighborhood of o^2 in a finite z interval, exits the neighborhood to pass through a neighborhood of o^1 in a later finite z interval before converging to o^2 as $z \to \infty$, see Fig. 2. A similar definition applies to 1-front heteroclinic orbits from o^2 to o^1. An n-front heteroclinic orbit is similar to a 1-front heteroclinic orbit except that it repeats the process of passing through the neighborhoods of o^1 and o^2 n times before converging to its limit as $z \to \infty$.

Many researchers have studied the existence of simple fronts and pulses of the FitzHugh-Nagumo equations, see e.g. [18, 4, 22, 12, 16]. Deng [8] discovered the n-fronts from the work of [22] and subsequently proved their existence for the FitzHugh-Nagumo equations. The n-front's stability was also conjectured in [8] that was later proved by Sandstede [23], who adopted an existence method somewhat different from [8]. The stability problem was also considered by Nii in his work [19].

The existence of n-fronts from [8] was established as bifurcations from a double twisted heteroclinic loop. The structure of "twist" deals with the orientations of individual heteroclinic orbits with respect to the loop, see Fig. 1, **H5**, Fig.4 of Sec.2. A heteroclinic loop bifurcation is generically a codimension-2 problem, that

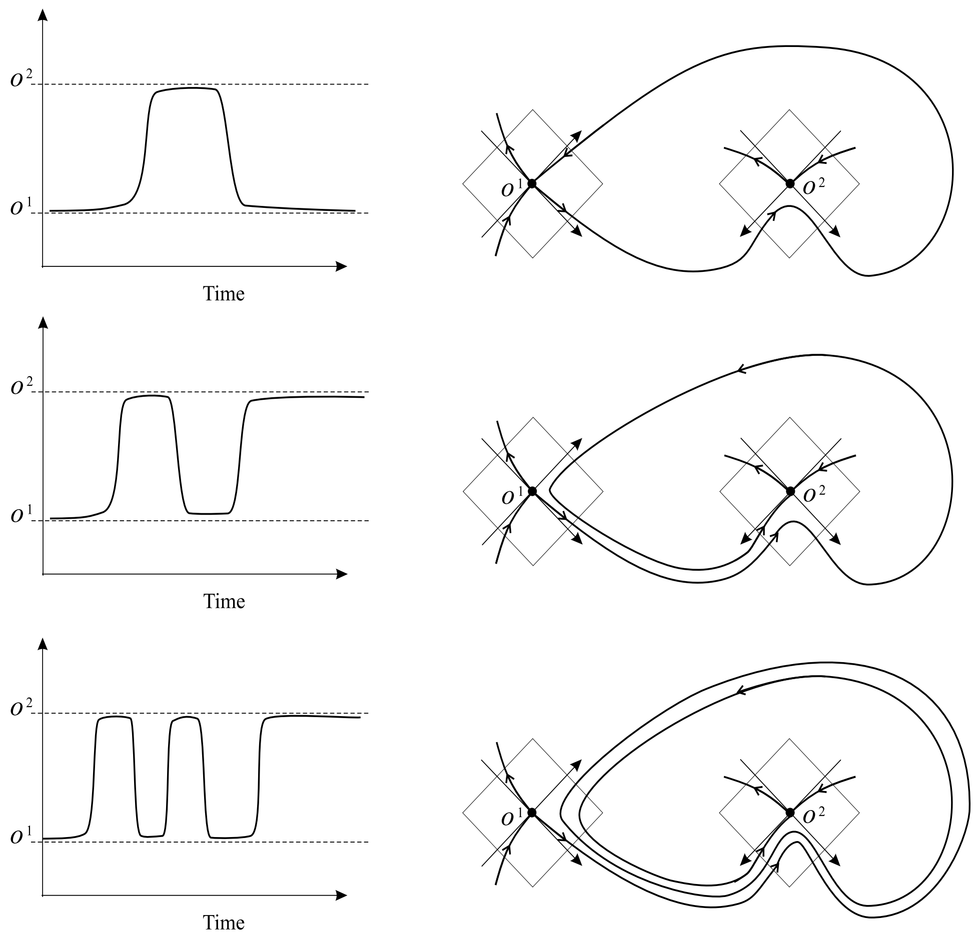

FIGURE 2. A bifurcated pulse homoclinic orbit(top), a 1-front heteroclinic orbit from o^1 to o^2(middle), and a 2-front heteroclinic orbit(bottom).

is, one only needs two generic parameters to unfold all possible perturbed structures near the heteroclinic loop point. One parameter is needed to induce a transversal breaking of each heteroclinic orbit in the loop. The FitzHugh-Nagumo equations (1.2) of the travelling coordinate frame possess three parameters: γ, ϵ, and the propagation speed c. The parameter ϵ is special in that it is asymptotically small: $0 < \epsilon \ll 1$. It is exactly this singular perturbation nature that allows [8] to show that Eq. (1.2) have a double twisted heteroclinic loop for any $0 < \epsilon \ll 1$ small and the parameters γ and c are two generic unfolding parameters. More specifically, the geometric theory for singular perturbations of ([10]) was used to reduce the problem from 3-dimension to 2-dimension when $\epsilon = 0$, and to obtain a "singular" double twisted heteroclinic loop at some parameter value (γ^*, c^*). The geometric theory of singular perturbations was used again to show that the singular loop persists for $0 < \epsilon \ll 1$ at some continued parameter values $(\gamma^*(\epsilon), c^*(\epsilon))$. Then by an earlier

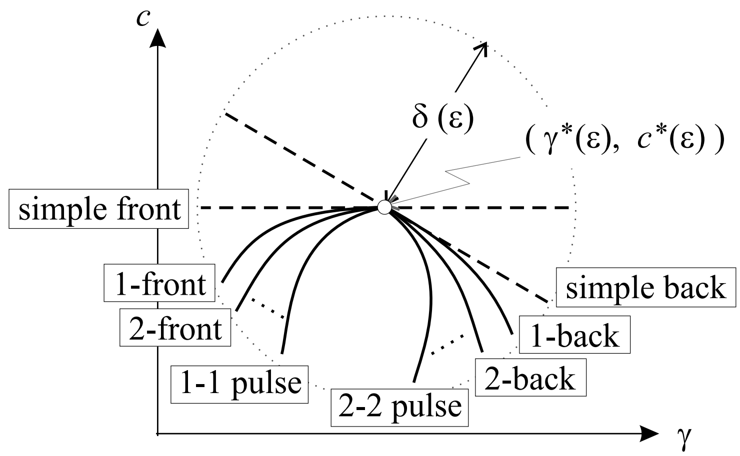

FIGURE 3. Bifurcation unfolding diagram at the twisted heteroclinic loop point $(\gamma^*(\epsilon), c^*(\epsilon))$ for the FitzHuge-Nagumo equations ([8]). "Simple front", "n-fronts" mean fronts from o^1 to o^2, and "simple back", "n-back" mean fronts from o^2 to o^1. "i-i pulse" means pulse from o^i to o^i.

nonsingular perturbation result of [7] it was concluded that for each $0 < \epsilon \ll 1$ fixed there exists a neighborhood of size $\delta(\epsilon) > 0$ such that the bifurcation diagram shown in Fig. 3 holds. Note that for each $n \geq 1$ there is a curve bifurcating from $(\gamma^*(\epsilon), c^*(\epsilon))$ in the γc-parameter space on which an n-front heteroclinic orbit exists from o^i to o^j. The size of the neighborhood depends on the value of ϵ in the way that the neighborhood radius $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. The same ϵ-dependence is also apparent in [23]. Interesting enough there is no such ϵ-dependence for the pulse bifurcation branches from earlier works, e.g., [4, 18, 22]. Thus the question is: can all the n-front bifurcation branches be continued to an ϵ-independent neighborhood of $(\gamma^*(\epsilon), c^*(\epsilon))$ like their pulse counterparts? The answer is Yes.

Our method departs from [8] and [23] in that instead of converting the singularly perturbed problem to a regular one for each $0 < \epsilon \ll 1$, we attack the problem directly for the range $0 < \epsilon \ll 1$. In this paper, however, we do not intend to give a complete treatment to the problem. Instead we want to expound in a more transparent way the key elements and ingredients for the uniformness of $\delta(\epsilon) \geq \delta_0 > 0$ as $\epsilon \to 0^+$. To do this we will substitute heuristic and in various places simplistic assumptions and arguments for a full-fledged proof. Justifications for doing so are two-folded: all the substitutive arguments can be made rigorous by various existing techniques ([5, 6, 14, 3]) and the full details can be found in [2].

The main result is presented in Sec. 2. The proof is given in Sec. 3. It is treated twice, progressively more realistic and complex. The first is done in $\mathbb{R}^2$ with various simplistic assumptions, some of which are not realistic. The second is done in $\mathbb{R}^3$ with realistic hypotheses. The sole purpose of treating the 2-dimensional problem is to introduce a skeleton proof for the real problem which has to be 3- or higher-dimensional. Each treatment highlights a different set of issues, ideas, and methods involved. It is our belief that ideas and techniques developed in this paper should prove useful in treating other singularly perturbed problems in applications.

2. The Main Result

We consider equations of the form

$$\dot{v} = X^\epsilon(v, \boldsymbol{\mu}), \tag{2.3}$$

where $v \in \mathbb{R}^3$, $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$, $0 \le \epsilon \ll 1$, $\dot{v} = \frac{dv}{dt}$ and X is C^∞ in all of its arguments. $\mathbb{R}^2_{loc}$ is a neighborhood of the origin in the parameter space. We place the singular parameter ϵ as a superscript to emphasis that it acts differently from parameter $\boldsymbol{\mu}$. We make the following assumptions **H1**–**H5** about Eq. (2.3). See Fig. 4 for a geometric representation of these assumptions.

First we assume that Eq. (2.3) has two 1-dimensional normally hyperbolic manifolds.

H1 Assume that there exist two compact connected 1-dimensional manifolds, K^i, $i = 1, 2$, such that

$$K^i(\boldsymbol{\mu}) \subset \{X^0(v, \boldsymbol{\mu}) = 0\},$$

(we consider K^i as being parameterized by $\boldsymbol{\mu}$). Furthermore, assume that for each $p \in K^i$, $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$, and $0 \le \epsilon \ll 1$, the spectrum of the matrix

$$D_v X^\epsilon(p, \boldsymbol{\mu}), \tag{2.4}$$

satisfies

$$\begin{gathered} -\lambda^{s,i}(p, \boldsymbol{\mu}, \epsilon) < -\lambda^{c,i}(p, \boldsymbol{\mu}, \epsilon) \le 0 < \lambda^{u,i}(p, \boldsymbol{\mu}, \epsilon), \\ \lambda^{c,i}(p, \boldsymbol{\mu}, 0) = 0, \text{ and } -\tfrac{\partial \lambda^{c,i}}{\partial \epsilon}(p, \boldsymbol{\mu}, 0) < 0. \end{gathered}$$

Assume also that for $0 < \epsilon$ there exists only one equilibrium point $o^i(\boldsymbol{\mu}, \epsilon)$, which is smooth in $(\boldsymbol{\mu}, \epsilon)$, satisfying $o^i(\boldsymbol{\mu}, 0) \in K^i$. Finally, we assume that K^i may be given as the graph of a single function over one of the variables.

By continuity of the eigenvalues with respect to all of their arguments, and the compactness of K^i, we may assume as a result of **H1** that there exist positive constants λ^s and λ^u such that

$$\begin{aligned} 0 \le \lambda^{c,i}(p, \boldsymbol{\mu}, \epsilon) \ll \lambda^s \le \lambda^{s,i}(p, \boldsymbol{\mu}, \epsilon) \\ 0 \le \lambda^{c,i}(p, \boldsymbol{\mu}, \epsilon) \ll \lambda^u \le \lambda^{u,i}(p, \boldsymbol{\mu}, \epsilon), \end{aligned}$$

for all $0 \le \epsilon \ll 1$, $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$, and $p \in K^i$. Also the following geometric structure for the flow of Eq. (2.3) in a neighborhood of K^i holds by **H1** and by the geometric theory of singular perturbation of Fenichel [10]. More specifically, if we extend Eq. (2.3) trivially by $\dot{\epsilon} = 0$, then for $|\epsilon| \ll 1$ and $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$, there exists a neighborhood of $K^i \times \{\epsilon\}$ such that in this neighborhood, there exists a (1+1)-dimensional center manifold, $W^{c,i}$ containing $K \times \{0\}$, a (2+1)-dimensional center-stable manifold, $W^{cs,i}$, and a (2+1)-dimensional center-unstable, $W^{cu,i}$, such that $W^{cs,i} \cap W^{cu,i} = W^{c,i}$. The extra dimension refers to the trivial dimension $\dot{\epsilon} = 0$. Furthermore, $W^{cs,i}$ and $W^{cu,i}$ are foliated by families of 1-dimensional manifolds parameterized by points of W^c, denoted by $\{F^s_\epsilon(p) : (p, \epsilon) \in W^{c,i}\}$ and $\{F^u_\epsilon(p) : (p, \epsilon) \in W^{c,i}\}$, respectively. All of the invariant manifolds above are parameterized by $\boldsymbol{\mu}$. Since planes of constant ϵ are invariant and intersect $W^{c,i}$, $W^{cs,i}$ and $W^{cu,i}$ transversely, we may consider these manifolds as 1-dimensional,

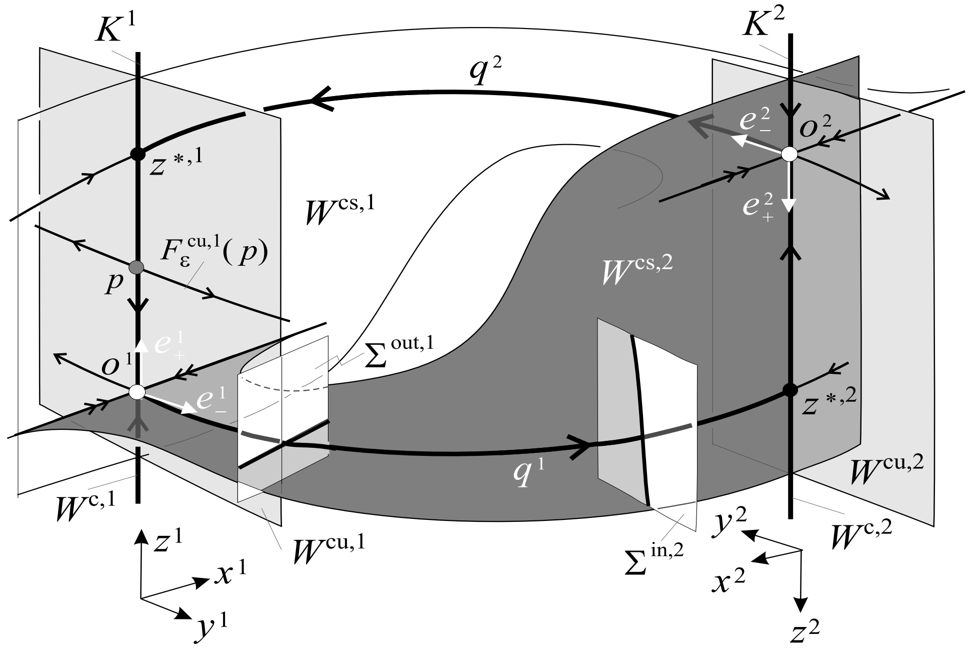

FIGURE 4. A singular twisted heteroclinic loop in $\mathbb{R}^3$, together with various invariant manifolds, foliations, cross sections, and local coordinates that are associated with the loop.

2-dimensional and 2-dimensional, respectively, and parameterized by ϵ and $\boldsymbol{\mu}$. We suppress the dependence of these manifolds on their parameters.

Next we assume that there is a singular heteroclinic loop for at $\epsilon = 0$.

H2 When $\epsilon = 0$ and $\boldsymbol{\mu} = (0, 0)$, Eq. (2.3) has heteroclinic orbits $q^i(t)$, connecting $o^i \in K^i$ to K^j, $i \neq j$, where $o^i \in K^i$ is as in **H1**. That is, there exist $z^{*,j} \in K^j$ such that

$$\lim_{t\to-\infty} q^i(t) = o^i, \ \lim_{t\to\infty} q^i(t) = z^{*,j}.$$

We also assume

$$o^j \neq z^{*,j}.$$

Note that the union of the orbits of q^i, the limiting points $o^i, z^{*,i}$, and the segments between $z^{*,i}$ and o^i on K^i defines what we call a singular heteroclinic loop, denoted by Γ^0. We assume Γ^0 is non-degenerate in the following sense.

H3 For $\boldsymbol{\mu} = (0, 0)$ assume that $q^i(t)$ converges along the leading stable direction of $z^{*,j}$ and the leading unstable direction of o^i as $t \to +\infty$ and $t \to -\infty$, respectively. We also assume that the intersection $W^{cs,j} \cap W^{cu,i}$ is transversal at $q^i(t)$.

Remark 2.1: The transversal intersection is generic. By the λ-lemma ([6]) this transversal intersection implies the following strong inclination property:

$$\lim_{t\to-\infty} T_{q^i(t)} W^{cs,j} = T_{o^i} W^u(o^i) + T_{o^i} W^{ss}(o^i)$$

where $W^{ss}(o^i)$ is the local strong stable manifold of o^i, and T_pW is the tangent space of a given manifold at a point p. $\Diamond$

The non-degeneracy hypothesis **H3** guarantees the existence of the Melnikov functions (see [20, 17]). Thus, generically, we assume the following condition.

H4 Let $\mathcal{M}^i(\boldsymbol{\mu}, \epsilon)$ be the Melnikov function for the separation of $W^u(o^i)$ and $W^{cs,j}$. Then we assume the vectors $D_{\boldsymbol{\mu}}\mathcal{M}^i((0,0),0) \in \mathbb{R}^2$, $i = 1, 2$, are linearly independent, and in particular non-zero.

Remark 2.2: The assumption that $D_{\boldsymbol{\mu}}\mathcal{M}^i((0,0),0), i = 1, 2$ are linearly independent implies by the Rank Theorem or the Implicit Function Theorem that $Q^i := \{\mathcal{M}^i(\boldsymbol{\mu}, \epsilon) = 0\}$ define two 2-dimensional surfaces in the parameter space $\mathbb{R}^2_{loc} \times \{\epsilon\}$ on which $W^u(o^i) \cap W^{cs,j} \neq \emptyset$. This in fact implies that for $0 < \epsilon \ll 1$, $q^i(t)$ is continued as a heteroclinic orbit from o^i to o^j. To see this recall from **H1** that

$$-\lambda^{c,i}(o^j, \boldsymbol{\mu}, \epsilon) < 0, \text{ for } 0 < \epsilon,$$

and o^j is the only equilibrium on $W^{c,j}$ for $0 < \epsilon$. Because the center eigenvalue shifts to the left of 0 for $0 < \epsilon$, the center-stable manifold $W^{cs,j}$ of K^j becomes the 2-dimensional stable manifold of o^j. Thus any flow through $F^{s,j}_\epsilon(p)$ for $p \neq o^j$, $p \in W^{cs,j}$ converges forward in time to o^j along the weaker attracting direction $W^{c,j}$. Since $W^u(o^i) \cap W^{cs,j} \neq \emptyset$ and $W^u(o^i)$ lies in a stable foliation $F^{s,j}_\epsilon(p)$ for some point p near $z^{*,j} \neq o^j$, we may assume by the hypotheses **H1**–**H4** that $q^i(t)$ is continued as a heteroclinic orbit from o^i to o^j for $(\boldsymbol{\mu}, \epsilon) \in Q^i$. The fact that $q^i(t)$ approaches o^j along the center direction $W^{c,j}$ because of the assumption $z^{*,j} \neq o^j$ implies that the continued heteroclinic orbit $q^i(t)$ is non-degenerate ([10, 7, 8, 17]).

The linear independence of $D_{\boldsymbol{\mu}}\mathcal{M}^i((0,0),0)$ from **H4** also implies that these two surfaces $Q^i, i = 1, 2$ cross transversely in a neighborhood of the origin and the intersection is necessarily a 1-dimensional curve in $\mathbb{R}^2_{loc} \times \{\epsilon\}$ parameterized by $0 \leq \epsilon$. Denote the parameterized intersection in $\boldsymbol{\mu}$ by $(\tilde{\mu_1}(\epsilon), \tilde{\mu_2}(\epsilon))$. Obviously the corresponding heteroclinic orbits $q^i(t)$ for $\boldsymbol{\mu} = (\tilde{\mu_1}(\epsilon), \tilde{\mu_2}(\epsilon))$ give rise to a heteroclinic loop denoted by Γ^ϵ which can be viewed as the continuation of Γ^0. Because each q^i is non-degenerate, the loop Γ^ϵ is non-degenerate as well.

Finally, the linear independence of $D_{\boldsymbol{\mu}}\mathcal{M}^i((0,0),0)$ from **H4** also implies that

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \epsilon \end{pmatrix} \to \begin{pmatrix} \mathcal{M}^1(\boldsymbol{\mu}, \epsilon) \\ \mathcal{M}^2(\boldsymbol{\mu}, \epsilon) \\ \epsilon \end{pmatrix},$$

defines a C^r change of parameters. Hence from now on we will assume the new parameters for which

$$Q^i = \{\mu_i = 0\}.$$

It is important to note that under this change of parameters, the Melnikov distance functions are independent of ϵ, a key property we will use later in our bifurcation arguments that hold uniformly in $0 < \epsilon \ll 1$. Also, because of this change of parameters we will assume from now on that the parameter region $\mathbb{R}^2_{loc}$ is explicitly independent of ϵ. $\Diamond$

Our last assumption is that Γ^ϵ is a double twisted heteroclinic loop in the following sense (see Fig. 4).

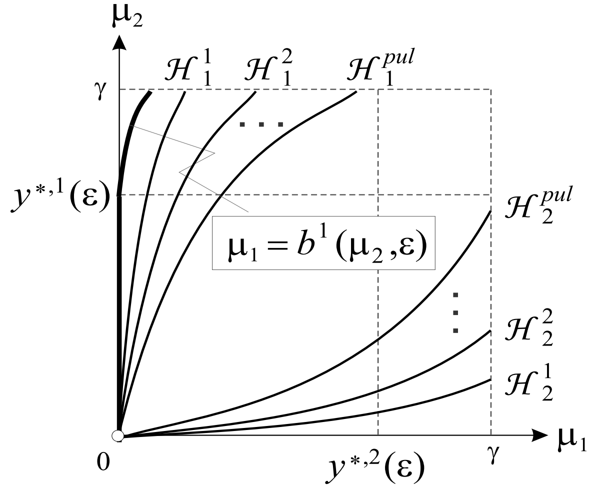

FIGURE 5. Bifurcation diagram in parameters (μ_1, μ_2). Definitions for $y^{*,1}, b^1$ are given and used later in Sec.3.

H5 Let Γ^ϵ be the continuation of Γ^0 as from the preceding paragraphs. Let e_-^j be the principal unstable eigenvector at o^j pointing to the same side of $W^{cs,j}$ as $q^j(t)$ does, that is,

$$e_-^j = \lim_{t \to -\infty} \frac{q^j(t) - o^j}{||q^j(t) - o^j||}.$$

Let e_+^i be the unit center eigenvector at o^i defined as

$$e_+^i = \lim_{t \to \infty} \frac{q^j(t) - o^i}{||q^j(t) - o^i||}.$$

For $\epsilon > 0$ and fixed, this is the principal, or weaker, stable eigenvector. We assume that e_+^i and e_-^j point to opposite sides of $T_{q^i(t)} W^{cs,j}$ as $t \to -\infty$ and $t \to \infty$, respectively.

We are now in a position to state our main result (see Fig. 5 for an illustration of the bifurcation diagram).

Theorem 2.3. *Assume that* **H1**–**H5** *are satisfied by Eq. (2.3). Then there exist small numbers* $0 < \epsilon_0 \ll 1$ *and* $0 < \gamma$*, independent of each other, such that for any* $r \geq 1$*, the following hold true (up to a* C^r *change of variables in* $\mathbb{R}^2_{loc}$*):*

(i) *There exists a unique function* $\mathcal{H}_1^{pul}(\mu_2, \epsilon)$*, defined over* $0 < \epsilon \leq \epsilon_0$ *and* $0 \leq \mu_2 \leq \gamma$*, such that for parameter values*

$$(\mathcal{H}_1^{pul}(\mu_2, \epsilon), \mu_2) \in \mathbb{R}^2_{loc} \text{ with } 0 < \mu_2 \leq \gamma,$$

Eq. (2.3) has a homoclinic pulse of o^1 *around* o^2*. Moreover we have that* $\mathcal{H}_1^{pul}(0, \epsilon) = 0$*. A similar statement holds made for a unique function* $\mu_2 = \mathcal{H}_2^{pul}(\mu_1, \epsilon)$*, defined on* $0 < \epsilon \leq \epsilon_0$ *and* $0 \leq \mu_1 \leq \gamma$*, and a homoclinic pulse of* o^2 *around* o^1*. Also, both curves* $\mu_j = \mathcal{H}_j^{pul}(\mu_i, \epsilon)$ *are smooth.*

(ii) *For each natural number $n \geq 1$, there exists a function $\mathcal{H}_1^n(\mu_2, \epsilon)$, defined over $0 < \epsilon \leq \epsilon_0$ and $0 \leq \mu_2 \leq \gamma$, such that for parameter values*

$$(\mathcal{H}_1^n(\mu_2, \epsilon), \mu_2) \in \mathbb{R}^2_{loc} \text{ with } 0 < \mu_2 \leq \gamma,$$

Eq. (2.3) has an n-front heteroclinic orbit from o^1 to o^2. Moreover we have that $\mathcal{H}_1^n(0, \epsilon) = 0$, and for each fixed $0 < \epsilon \leq \epsilon_0$ and $0 < \mu_2 \leq \gamma$, these functions have the ordering

$$0 < \cdots < \mathcal{H}_1^n(\mu_2, \epsilon) < \mathcal{H}_1^{n+1}(\mu_2, \epsilon) < \cdots < \mathcal{H}_1^{pul}(\mu_2, \epsilon).$$

A similar statement can be made for the existence of n-front heteroclinic orbits from o^2 to o^1 for parameter values satisfying $\mu_2 = \mathcal{H}_2^n(\mu_1, \epsilon), 0 < \epsilon \leq \epsilon_0$ and $0 \leq \mu_1 \leq \gamma$.

(iii) *$\mathcal{H}_1^{pul}$, $\mathcal{H}_2^{pul} \to 0^+$ as $\epsilon \to 0^+$ uniformly in $\boldsymbol{\mu}$, and hence, so do all of the above mentioned functions by the ordering given in (ii).*

Remark 2.4: Theorem A of [7] states that the curves given by $\mathcal{H}_i^{pul}$ and $\mathcal{H}_i^n$ are asymptotically tangent to the μ_j axis, as $\mu_j \to 0^+$, $i, j = 1, 2$ and $i \neq j$.

Hypotheses **H1**–**H5** are satisfied by the FitzHugh-Nagumo equations as shown in [8, 23, 2], and hence the conclusions of this theorem follow.$\diamondsuit$

3. Proofs

Our uniform arguments for $\delta(\epsilon) \geq \delta_0 > 0$ for all $0 < \epsilon \ll 1$ are heuristic and, in various places, simplistic, presented in two skeletal proofs, progressively more realistic. The first one strips off all nonessential elements to focus on the uniform continuation issue, using a non-realistic 2-dimensional model. The second one uses a simplistic but nonetheless realistic 3-dimensional model to show how the key uniform arguments for the 2-dimensional model still hold.

3.1. **The 2-Dimensional Case.**

A common approach to all the proofs, skeletal or full-fledged, is to derive the n-front bifurcation equations from the Poincaré return map of the heteroclinic loop Γ^0 and to obtain the bifurcation branches as solutions to the bifurcation equations. Also common to this approach is to decompose the Poincaré map into flow-induced maps near the slow manifolds K^i and flow-induced maps near the heteroclinic orbits q^i connecting a neighborhood of K^i to a neighborhood of K^j.

3.1.1. *Simplistic Assumptions.*

The assumption that a singular perturbed heteroclinic loop occurs in dimension 2 is not realistic because there are not enough dimensions to accommodate such singular structures. On the other hand this assumption is not without justification. This is because that for a non-singularly perturbed situation, a 2-dimensional center manifold of the loop Γ^0 exists ([23]) which allows a dimensional reduction to a 2-dimensional problem. Although such smooth manifold is not expected for singularly perturbed equations, an analogous reduction in some less ideal form is conjectured to exist. We will see shortly that instead of actually assuming the existence of singular heteroclinic orbits between K^i and K^j, we assume the existence of certain diffeomorphisms from a neighborhood of K^i to a neighborhood of K^j.

The second simplistic assumption is to assume that the equation near the slow manifolds K^i is linearized as follows. That is, in a square neighborhood S^i of o^i there exists local coordinates such that Eq. (2.3) is of the form

$$\begin{aligned} \dot{x} &= -\epsilon x \\ \dot{y} &= \lambda_i y, \quad 0 \le \epsilon < \lambda_i. \end{aligned} \tag{3.5}$$

Note that in S^i we have $o^i = (0,0)$, and K^i is the x-axis when $\epsilon = 0$. We also have that for $0 < \epsilon$, the local stable manifold of o^i, $W^s(o^i)$, is the x-axis and the local unstable manifold of o^i, $W^u(o^i)$, is the y-axis. We orient the coordinates in S^i so that $e^i_- = (0,1)^T$ and $e^i_+ = (1,0)^T$. Since the principal stable eigendirection for the nonlinear case is precisely the singular (center) direction, the inclusion of $\dot{x} = -\epsilon x$ captures the singular nature of the problem. We note that such linearization is not always true but can be remedied by various treatments (see more discussion later).

3.1.2. *Poincaré Maps.* We define the Poincaré return map by first defining the Poincaré sections in S^i as follows

$$\begin{aligned} \Sigma^{out,i} &:= \{(x,1) : |x| \le 1\} \\ \Sigma^{in,i} &:= \{(1,y) : |y| \le 1\}. \end{aligned}$$

These sections are transversal to the flow near q^i and q^j, respectively. We place local coordinates on these sections by projecting the coordinates in S^i onto them, see Fig. 1 and Fig. 6.

For a non-singular heteroclinic loop Γ^0 such as depicted in Fig. 1, we can define a flow-induced global map $\Pi^{far,i}$ from $\Sigma^{out,i}$ to $\Sigma^{in,j}$. This map is defined in a neighborhood of q^i and it is a diffeomorphism. Since there is not a singular heteroclinic loop in $\mathbb{R}^2$, we will simply assume the existence of such a global map satisfying all defining properties in a non-singular 2-dimensional case as well as in a realistic, singular 3-dimensional case. That is, we assume a diffeomorphism

$$\Pi^{far,i} : \{|x| < \eta\} \times \mathbb{R}^2_{loc} \times \{\epsilon\} \to \Sigma^{in,j}, \tag{3.6}$$

for $0 < \eta < 1$ and $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$. We assume that it is independent of $\mu_j, j \ne i$, ϵ, and given by

$$\Pi^{far,i}(x, \mu_i) = \mu_i - c_i x. \tag{3.7}$$

In particular, when $x = 0$ the map coincide with the corresponding Melnikov function implied by **H4**, and $\mu_i = 0$ corresponds to the persistence of the heteroclinic orbit q^i. Finally, our twist assumption on q^i requires that for $x > 0$, $\Pi^{far,i}(x,0) < 0$, and hence the minus sign in front of the positive constant c_i, see Fig. 6.

The domain of definition for the uniform existence interval $[0,\gamma]$ for $\mathcal{H}^n_j(\mu_i,\epsilon)$ with $\mu_i \in [0,\gamma]$ and for all n will be obtained by some careful analysis on both the global and local maps. It results from our requiring the Poincaré map to be well-defined. More precisely, for $\Pi^{far,i}$ to map $\Sigma^{out,i}$ into $\Sigma^{in,j}$ for all $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$ we need

$$-1 \le \mu_i - c_i\eta \le \Pi^{far,i}(x,\mu_i) \le \mu_i + c_i\eta \le 1,$$

as $|x| \le \eta < 1$, see Fig. 6. This automatically put a constraint on the parameter region in $\boldsymbol{\mu}$. So we must require that

$$|\mu_i| \le 1 - c_i\eta,$$

which gives rise to the ϵ-independent constant γ as follows,

$$\gamma = \min\{1 - c_1\eta,\ 1 - c_2\eta\} \text{ so that } \mathbb{R}^2_{loc} := \{|\boldsymbol{\mu}| \le \gamma\}. \tag{3.8}$$

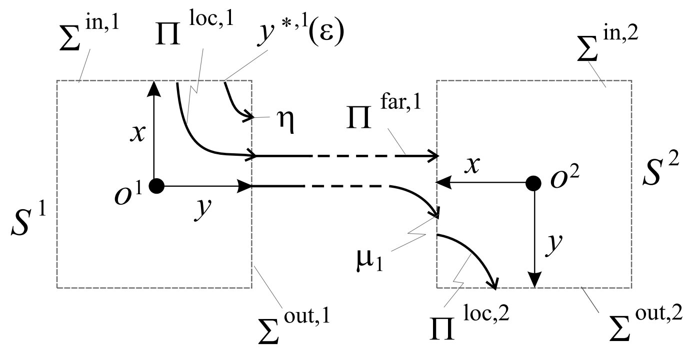

FIGURE 6. The local coordinates and the heteroclinic orbit q^1.

We point out that for consistency we require the constant η as in the domain of $\Pi^{far,i}$ to satisfy

$$0 < \eta < 1 \text{ and } 1 - c_i\eta > 0,$$

which we assume throughout.

Next we define the flow-induced local maps,

$$\Pi^{loc,i} : \sigma^{in,i} \subset \Sigma^{in,i} \to \Sigma^{out,i}, \tag{3.9}$$

where $\sigma^{in,i}$ is the domain of definition. These maps are obtained explicitly from solutions to (3.5),

$$x(t) = x_0 e^{-\epsilon t}, \quad y(t) = y_0 e^{\lambda_i t},$$

for $x_0, y_0 > 0$. Let $(x_0, y_0) \in \Sigma^{in,i}$ with $x_0 = 1$ and τ be the time of flight to reach $\Sigma^{out,i}$, i.e., $(x(\tau), y(\tau)) \in \Sigma^{out,i}$. Then

$$1 = y_0 e^{\lambda_i \tau}.$$

Solving τ in terms of y_0 and substituting it into $x(\tau)$ give rise to the local maps

$$\Pi^{loc,i} : y_0 \to y_0^{\epsilon/\lambda_i}.$$

In order to compose the local map $\Pi^{loc,i}$ with the global map $\Pi^{far,i}$ we need to require the image of the local map be contained in the domain $\Sigma^{out,i} = \{x : |x| \leq \eta\}$ of the global map. This requirement in turn gives rise to the local map's domain of definition $\sigma^{in,i}$. More specifically, $0 < y_0 \in \sigma^{in,i}$ iff

$$\Pi^{loc,i}(y_0, \epsilon) = y_0^{\epsilon/\lambda_i} \leq \eta.$$

Solving for the largest y_0 satisfying the above equation leads to the domain of definition as follows

$$\sigma^{in,i} := \{0 < y_0 \leq y^{*,i}(\epsilon)\} \text{ with } y^{*,i}(\epsilon) := \eta^{\lambda_i/\epsilon}.$$

See Fig. 6 for an illustration.

A few remarks about the local maps are in order. As $(1, 0) = W^s(o^i) \cap \Sigma^{in,i}$ the flow through this point converges to o^i forward in time. Since o^i is not in $\Sigma^{out,i}$, $(1, 0)$ is not in $\sigma^{in,i}$. It is at the boundary because it corresponds to $y_0 = 0$. So $\Pi^{loc,i}$ can be extended in a C^0 fashion to $y_0 = 0$ with $\Pi^{loc,i}(0) = 0$. This corresponds to an infinite time of flight, connecting $(1, 0)$ to $(0, 1)$. This map becomes singular at

$\epsilon = 0$. Also we note two important facts about the right end point $y^{*,i}(\epsilon)$ of the local map's definition interval $\sigma^{in,i}$:

(1) $y^{*,i}(\epsilon) \to 0$ as $\epsilon \to 0^+$ (again, the singular nature of $\Pi^{loc,i}$).
(2) As $y_0 \to y^{*,i}(\epsilon)$ we have $\Pi^{loc,i}(y_0, \epsilon) = y_0^{\epsilon/\lambda_i} \to \eta$ monotonically, in particular, $\Pi^{loc,i}(y^{*,i}(\epsilon), \epsilon) = \eta$.

These two properties will be used later for the uniform continuation argument.

In what is to follow we will also use the composition

$$\Pi^{loc,j} \circ \Pi^{far,i}(0, \mu_i) = \Pi^{loc,j}(\mu_i, \epsilon) = \mu_i^{\epsilon/\lambda_i},$$

as a way to keep track of the unstable manifold $W^u(o^i)$ for $\mu_i > 0$. We will require that as we sweep $\mu_i = \Pi^{far,i}(0, \mu_i)$ through the parameter interval $[0, \gamma]$, the image of $\Pi^{far,i}$ of 0 sweep through the entire domain of definition $\sigma^{in,j} = \{0 \leq y \leq y^{*,j}(\epsilon)\}$ of $\Pi^{loc,j}$. This is guaranteed by the first property above by choosing $\epsilon_0 > 0$ small enough such that for every $0 \leq \epsilon \leq \epsilon_0$ we have $y^{*,j}(\epsilon) < \gamma$.

3.1.3. *Homoclinic Bifurcations.* In this section we discuss the bifurcation of Γ^0 to a homoclinic orbit of o^1 passing through S^2. The discussion of a homoclinic orbit of o^2 passing through S^1 is similar. For each $0 < \epsilon \leq \epsilon_0$ fixed, we need to obtain a curve, $\mu_1 = \mathcal{H}_1^{pul}(\mu_2, \epsilon)$, in the parameter space $\mathbb{R}^2_{loc}$ for every $0 < \mu_2 \leq \gamma, 0 < \epsilon \ll 1$ such there is a pulse homoclinic orbit of o^1. The existence of such an orbit is equivalent to finding

$$\boldsymbol{\mu} \in \mathbb{R}^2_{loc}, \text{ with } y_1 = 0$$

that satisfy

$$\begin{array}{rcl} \Pi^{far,1}(0, \mu_1) & = & y_0 \\ \Pi^{far,2}(y_0^{\epsilon/\lambda_2}, \mu_2) & = & y_1 = 0 \end{array} \tag{3.10}$$

or equivalently

$$\begin{array}{rcl} \mu_1 & = & y_0 \\ \mu_2 - c_2 y_0^{\epsilon/\lambda_2} & = & 0. \end{array}$$

For this 2-dimensional simplistic model, one can solve explicitly for μ_1 in μ_2 *formally* and argue that the domain of definition is $\mu_2 \in (0, \gamma]$. But for a full-fledged proof such explicit expression is not available. The argument is based on a shooting method instead. It consists of two steps:

Step 1: Find a region in the parameter region $\mathbb{R}^2_{loc}$ in which the compositions defined by the equations (3.10) are well-defined. It is straightforward to check that this is the case because

$$0 \leq \mu_1 = y_0 \leq y^{*,2}(\epsilon) \text{ and } 0 \leq \mu_2 \leq \gamma.$$

Step 2: In this step we need to show that in the region defined above, there exists a value of μ_1^- such that $\mu_2 - c_2(\mu_1^-)^{\epsilon/\lambda_2} = y_1^- < 0$, and there exists another value of μ_1^+ such that $\mu_2 - c_2(\mu_1^+)^{\epsilon/\lambda_2} = y_1^+ > 0$. The solution then follows from the Intermediate Value Theorem. We will also show that this value is unique, a key point to be used later.

For the shooting method to work, we first need to find the regions in $\Sigma^{out,2}$ on which $\Pi^{far,2}(x, \mu_2) > 0$ (< 0). To do this we first find x such that

$$\Pi^{far,2}(x, \mu_2) = \mu_2 - c_2 x = 0,$$

which gives rise to $x = \mu_2/c_2$. Because $\Pi^{far,2}$ is strictly decreasing in x, we find the needed regions as

$$(3.11) \qquad \begin{array}{lll} \text{if } 0 \le x < \mu_2/c_2 & \text{then} & \Pi^{far,2}(x,\mu_2) > 0, \\ \text{and if } \mu_2/c_2 < x < \eta & \text{then} & \Pi^{far,2}(x,\mu_2) < 0. \end{array}$$

In order for $\mu_2/c_2 \le \gamma/c_2 < \eta$ to hold we refine the definition of γ in (3.8) as follows

$$\gamma < \min\{c_i\eta, 1 - c_i\eta, i = 1, 2\},$$

and require that $|\boldsymbol{\mu}| \le \gamma$. Note that the definition of γ is independent of ϵ, a key point for the uniform argument. Note also that this choice of γ depends only the global maps $\Pi^{far,i}$. In the $\mathbb{R}^3$ model and the proof of [2], this independence of γ from ϵ comes from the regular perturbation nature of $\Pi^{far,i}$ in ϵ.

We now proceed with the actual shooting. Fix $0 < \epsilon \le \epsilon_0$ and $0 < \mu_2 < \gamma$. We have that $\mu_1 = y_0$. If $\mu_1^+ = 0$ then

$$x = y_0^{\epsilon/\lambda_2} = 0 < \mu_2/c_2$$

and, by (3.11) above, we have that

$$y_1^+ = \Pi^{far,2}(x, \mu_2) = \mu_2 > 0.$$

On the other hand, if $\mu_1^- = y_0 = y^{*,2}(\epsilon)$, then, by the definition of $y^{*,2}(\epsilon)$, we have

$$x = y_0^{\epsilon/\lambda_2} = \eta > \gamma/c_2 \ge \mu_2/c_2.$$

Hence, by (3.11),

$$y_1^- = \Pi^{far,2}(x, \mu_2) = \mu_2 - c_2\eta < 0.$$

As all of our maps are continuous in μ_1, we can conclude, by the Intermediate Value Theorem, that there exists a value of μ_1, between $\mu_1^+ = 0$ and $\mu_1^- = y^{*,2}(\epsilon)$, such that

$$y_1 = \Pi^{far,2}(\mu_1^{\lambda_2/\epsilon}, \mu_2) = 0.$$

This defines a function in $\mathbb{R}^2_{loc}$ denoted by $\mu_1 = \mathcal{H}_1^{pul}(\mu_2, \epsilon) \in (0, y^{*,2}(\epsilon))$. See Fig. 5 for an illustration of this argument. In fact, this curve is uniquely defined and we can calculate it from (3.10) as

$$\mu_1 = \mathcal{H}_1^{pul}(\mu_2, \epsilon) = (\mu_2/c_2)^{\lambda_2/\epsilon}.$$

We point out that as this function is unique, for $0 \le \mu_1 < \mathcal{H}_1^{pul}(\mu_2, \epsilon)$ we have $y_1 = \Pi^{far,2} \circ \Pi^{loc,1}(0, \mu_1) > 0$, and for $\mathcal{H}_1^{pul}(\mu_2, \epsilon) < \mu_1 \le y^{*,2}(\epsilon)$ we have $y_1 = \Pi^{far,2} \circ \Pi^{loc,1}(0, \mu_1) < 0$. A symmetrical property is listed below for a later reference.

Remark 3.1: For each $0 < \mu_1 < \gamma$ and $0 < \epsilon \le \epsilon_0$ fixed, there exists a curve $\mu_2 = \mathcal{H}_2^{pul}(\mu_1, \epsilon)$, between 0 and $y^{*,1}(\epsilon)$, such that

$$(3.12) \qquad \begin{array}{rcl} \Pi^{far,2}(0, \mu_2) & = & y_0 \\ \Pi^{far,1}(y_0^{\epsilon/\lambda_1}, \mu_1) & = & y_1 = 0. \end{array}$$

In particular, $\mu_2 > \mathcal{H}_2^{pul}(\mu_1, \epsilon)$ if and only if $y_1 = \Pi^{far,1} \circ \Pi^{loc,2}(0, \mu_2) < 0$.

We also have from above that $\mathcal{H}_1^{pul}$, $\mathcal{H}_2^{pul} \to 0^+$ as $\epsilon \to 0^+$ because $y^{*,i}(\epsilon) \to 0$ as $\epsilon \to 0^+$. Thus, for ϵ small, $\mathcal{H}_1^{pul}$ lies above $\mathcal{H}_2^{pul}$ in the μ_2 direction, see Fig. 5. With this in mind we define the set

$$\Lambda^0 := \{\boldsymbol{\mu} : 0 \le \mu_2 < \gamma,\ 0 \le \mu_1 \le \mathcal{H}_1^{pul}(\mu_2, \epsilon)\}.$$

This set will be used extensively in subsequent subsections for a shooting argument for the n-front bifurcation branches. The uniformness is to be derived by showing that the n-front curves $\mathcal{H}_1^n$ are wedged in the sector region Λ^0, connecting from Λ^0's cusp point $\boldsymbol{\mu} = 0$ to its boundary $\mu_2 = \gamma$. By what was said in the preceding paragraph, for any $\boldsymbol{\mu} \in \Lambda^0 \backslash (0,0)$ for which (3.12) is well-defined, $y_1 = \Pi^{far,1} \circ \Pi^{loc,2}(0, \mu_2) = \Pi^{far,1}(\mu_2^{\epsilon/\lambda_1}, \mu_1) < 0$ iff $\mu_2 > \mathcal{H}_2^{pul}(\mu_1, \epsilon)$. This is one of the key points for the uniform argument of the next two subsections.

We point out that comparing to what will follow on the n-front bifurcations it requires no special treatments on the uniform continuation of the homoclinic branches $\mathcal{H}_i^{pul}$. The argument essentially recaptures the same geometric idea used by [4]. $\Diamond$

3.1.4. *1-Front Heteroclinic Bifurcation.* For each $0 < \epsilon \leq \epsilon_0$ fixed, we need to find a 1-front heteroclinic curve $\mu_1 = \mathcal{H}_1^1(\mu_2, \epsilon)$ in $\mathbb{R}^2_{loc}$ for $\mu_2 \in (0, \gamma]$. We will look for $\mathcal{H}_1^1$ in the sector region Λ^0. The existence of a 1-front of o^1 to o^2 is equivalent to finding

$$\boldsymbol{\mu} \in \mathbb{R}^2_{loc} \text{ with } y_2 = 0$$

that satisfy the equations

$$\begin{array}{rcl} \Pi^{far,1}(0, \mu_1) & = & y_0 \\ \Pi^{far,2}(y_0^{\epsilon/\lambda_2}, \mu_2) & = & y_1 \\ \Pi^{far,1}(y_1^{\epsilon/\lambda_1}, \mu_1) & = & y_2 = 0, \end{array} \tag{3.13}$$

or equivalently

$$\begin{array}{rcl} \mu_1 & = & y_0 \\ \mu_2 - c_2 y_0^{\epsilon/\lambda_2} & = & y_1 \\ \mu_1 - c_1 y_1^{\epsilon/\lambda_1} & = & 0. \end{array}$$

To solve this system of equations for μ_1 in terms of μ_2, we utilize a similar two step shooting method as for the homoclinic branches.

Step 1: Our first concern is whether or not the compositions required by (3.13) are well-defined. Fix $0 < \mu_2 \leq \gamma$ and $0 < \epsilon \leq \epsilon_0$. If $\mu_1 = \mathcal{H}_1^{pul}(\mu_2, \epsilon)$ then we know, from the result in the previous section, that $0 < y_0 < y^{*,2}(\epsilon)$ and $y_1 = 0$. Hence, the composition is well-defined at this point. Also the first two equations are well defined for $\mu_1 \leq \mathcal{H}_1^{pul}(\mu_2, \epsilon)$ as demonstrated in the previous subsection for homoclinic bifurcation because such a point (μ_1, μ_2) lies in Λ^0. Recall in addition that $\mu_1 < \mathcal{H}_1^{pul}(\mu_2, \epsilon)$ iff $y_1 = \Pi^{far,2} \circ \Pi^{loc,1}(0, \mu_1) > 0$. So we only need to know how much smaller μ_1 has to be in order for the last equation to be defined. Hence the only condition left to consider is

$$0 \leq y_1 \leq y^{*,1}(\epsilon)$$

or

$$0 \leq \mu_2 - c_2 \mu_1^{\epsilon/\lambda_2} \leq y^{*,1}(\epsilon).$$

Solving the two inequalities above for $\mu_1 < \mathcal{H}_1^{pul}(\mu_2, \epsilon)$ yields

$$\mathcal{H}_1^{pul}(\mu_2, \epsilon) = \left(\frac{\mu_2}{c_2}\right)^{\lambda_2/\epsilon} > \mu_1 \geq b^1(\mu_2, \epsilon) := \begin{cases} 0 & , \quad 0 < \mu_2 \leq y^{*,1}(\epsilon) \\ \left(\frac{\mu_2 - y^{*,1}(\epsilon)}{c_2}\right)^{\lambda_2/\epsilon} & , \quad \mu_2 > y^{*,1}(\epsilon). \end{cases} \tag{3.14}$$

That is, (3.13) is well-defined for

$$\boldsymbol{\mu} \in \Lambda^1 := \{\boldsymbol{\mu} : \; b^1(\mu_2, \epsilon) \leq \mu_1 \leq \mathcal{H}_1^{pul}(\mu_2, \epsilon), 0 \leq \mu_2 \leq \gamma\} \subset \Lambda^0.$$

Fig. 5 shows a typical graph of b^1 in $\mathbb{R}^2_{loc}$ and the bounding sector Λ^1.
Remark 3.2:

(1) The need to define $b^1(\mu_2, \epsilon)$ for $\mu_2 > y^{*,1}(\epsilon)$ is the key point in showing that the domain of definition of $\mathcal{H}^1_1(\mu_2, \epsilon)$ is uniform in ϵ. To see this we recall the work of [7]. In it the author only used the set where a simple front exists, the μ_2-axis in our case, which corresponds to $b^1(\mu_2, \epsilon) = 0$ for $0 \le \mu_2 \le y^{*,1}(\epsilon)$. But $y^{*,1}(\epsilon) \to 0^+$ as $\epsilon \to 0^+$, which explains why the domain of definition for $\mathcal{H}^1_1(\mu_2, \epsilon)$ in [7] is ϵ-dependent.
(2) Since everything is continuous, we make an alternative definition for b^1 as

$$b^1(\mu_2, \epsilon) := \inf\{b : 0 \le b, \text{ and for } b \le \mu_1 \le \mathcal{H}^{pul}_1, \text{ we have } 0 \le y_1 \le y^{*,1}(\epsilon)\},$$

which is the definition that was used in the proof of [2]. From the first definition of b^1 we see that if $\mu_1 = b^1(\mu_2, \epsilon) > 0$, then $y_1 = y^{*,1}(\epsilon)$. To see that this follows from the second definition, we notice that for μ_1 on the boundary of (3.14), it must be either y_1 is 0 or $y^{*,1}(\epsilon)$. If $y_1 = 0$ we would have a pulse of o^1. Since this occurs only for $\mu_1 = \mathcal{H}^{pul}_1(\mu_2, \epsilon)$ which is unique for homoclinic orbit of o^1, it must follow that $y_1 = y^{*,1}(\epsilon)$.

$\lozenge$

Step 2: We now do the actual shooting. If $\mu_1^+ = \mathcal{H}^{pul}_1(\mu_2, \epsilon)$ then $y_1 = 0$ and hence $y_2^+ = \Pi^{far,1}(0, \mu_1^+) = \mu_1^+ > 0$. On the other hand we show that if $\mu_1^- = b^1(\mu_2, \epsilon)$ then $y_2^- < 0$. To do this we need to consider two cases: $b^1 > 0$ and $b^1 = 0$. For the first case, we have by the definition of b^1 that for $\mu_1^- = b^1(\mu_2, \epsilon) > 0$, $y_1 = y^{*,1}(\epsilon)$. Hence $\Pi^{loc,1}(y_1, \epsilon) = y_1^{\epsilon/\lambda_1} = \eta$. By (3.11), with the index 2 replaced by 1, we have $y_2^- = \Pi^{far,1}(\eta, \mu_1^-) < 0$. For the second case, when $b^1 = 0$, the last two equations of (3.13) are exactly (3.12), the bifurcation equation of a pulse of o^2. By Remark 3.1.3 we again have $y_2^- < 0$. The Intermediate Value Theorem gives the existence of a value of μ_1, between $\mu_1^- = b^1(\mu_2, \epsilon)$ and $\mu_1^+ = \mathcal{H}^{pul}_1(\mu_2, \epsilon)$, such that $y_2 = 0$. Choosing the first such point below $\mathcal{H}^{pul}_1(\mu_2, \epsilon)$ defines the function $\mathcal{H}^1_1(\mu_2, \epsilon)$. Furthermore, $0 < \mathcal{H}^1_1(\mu_2, \epsilon) < \mathcal{H}^{pul}_1(\mu_2, \epsilon)$ for $0 < \epsilon \le \epsilon_0$ and $0 < \mu_2 \le \gamma$. Notice that unlike $\mathcal{H}^{pul}_1$ this function may not be unique, which will not be needed in what follows. Also notice that by our choice of $\mathcal{H}^1_1(\mu_2, \epsilon)$, which is the first point where $y_2 = 0$ and below $\mathcal{H}^{pul}_1$, we have $y_2 > 0$ for $\mathcal{H}^1_1(\mu_2, \epsilon) < \mu_1 \le \mathcal{H}^{pul}_1(\mu_2, \epsilon)$.

3.1.5. *n-front Bifurcations.* We need to show that for the γ defined in the previous sections and $0 < \epsilon \le \epsilon_0$ there exists an n-front heteroclinic branch $\mu_1 = \mathcal{H}^n_1(\mu_2, \epsilon)$ defined for $0 \le \mu_2 \le \gamma$ with its graph in Λ^0, as shown in Fig. 5. We will do so by an inductive argument. Roughly speaking, we will show that $\mathcal{H}^1_1$ is used to define a curve b^2 so that b^2 and $\mathcal{H}^{pul}_1$ are used to bound a sector $\Lambda^2 \subset \Lambda^1$ in which $\mathcal{H}^2_1$ is found using b^2 and $\mathcal{H}^{pul}_1$ as shooting boundaries, and so on.

Similar to the 1-front case, we start with the equations for the n-fronts. Finding an n-front is equivalent to finding

$$\boldsymbol{\mu} \in \mathbb{R}^2_{loc}, \text{ with } y_{2n} = 0$$

that satisfy the equations

$$
\begin{array}{lcl}
\Pi^{far,1}(0,\mu_1) & = & y_0 \\
\Pi^{far,2}((y_0)^{\epsilon/\lambda_2},\mu_2) & = & y_1 \\
 & \vdots & \\
\Pi^{far,1}((y_{2n-1})^{\epsilon/\lambda_1},\mu_1) & = & y_{2n},
\end{array} \tag{3.15}
$$

or equivalently

$$
\begin{array}{lcl}
\mu_1 & = & y_0 \\
\mu_2 - c_2 y_0^{\epsilon/\lambda_2} & = & y_1 \\
 & \vdots & \\
\mu_1 - c_1 y_{2n-1}^{\epsilon/\lambda_1} & = & 0.
\end{array}
$$

To this end we utilize a similar shooting method again, only now we do it inductively.

We assume the following induction hypotheses hold true for $1 \le m \le n-1$:

(1) There exists a curve $\mathcal{H}_1^m(\mu_2,\epsilon) = \mu_1$, defined over $0 \le \mu_2 \le \gamma$ and $0 < \epsilon \le \epsilon_0$, such that (2.3) has an m-pulse heteroclinic orbit from o^1 to o^2.

(2) For $0 < \mu_2 \le \gamma$, we have the ordering

$$0 \le \mathcal{H}_1^{m-1}(\mu_2,\epsilon) < \mathcal{H}_1^m(\mu_2,\epsilon) < \mathcal{H}_1^{pul}(\mu_2,\epsilon).$$

Equality on the left only holds for $\mathcal{H}^0$ which we define to be the μ_2-axis, the set where a simple front occurs.

(3) The compositions defined in (3.15), with n replaced by m, are well-defined on the sets

$$\Lambda^m := \{\boldsymbol{\mu} : b^m(\mu_2,\epsilon) \le \mu_1 \le \mathcal{H}_1^{pul}(\mu_2,\epsilon), 0 \le \mu_2 \le \gamma\},$$

where

$$
\begin{array}{ll}
b^m(\mu_2,\epsilon) := \inf\{b : & \mathcal{H}_1^{m-1}(\mu_2,\epsilon) \le b, \text{ and} \\
 & 0 \le y_{2m-1} \le y^{*,1}(\epsilon) \text{ for } b \le \mu_1 \le \mathcal{H}_1^{pul}(\mu_2,\epsilon)\}.
\end{array}
$$

Furthermore, $b^m(\mu_2,\epsilon)$ satisfies the following properties:

(a) $b^{m-1}(\mu_2,\epsilon) < \mathcal{H}_1^{m-1}(\mu_2,\epsilon) \le b^m(\mu_2,\epsilon) < \mathcal{H}_1^{pul}(\mu_2,\epsilon)$ which implies

$$\Lambda^m \subset \Lambda^{m-1};$$

(b) $0 \le y_{2m-1} \le y^{*,1}(\epsilon)$ for $b^m(\mu_2,\epsilon) \le \mu_1 \le \mathcal{H}_1^{pul}(\mu_2,\epsilon)$;

(c) $y_{2m-1} = 0$ for $\mu_1 = \mathcal{H}_1^{pul}(\mu_2,\epsilon)$ and,

$$
y_{2m-1} = \begin{cases} y^{*,1}(\epsilon), & \text{if } \mu_1 = b^m(\mu_2,\epsilon) > \mathcal{H}_1^{m-1}(\mu_2,\epsilon) \\ \mu_2, & \text{if } \mu_1 = b^m(\mu_2,\epsilon) = \mathcal{H}_1^{m-1}(\mu_2,\epsilon). \end{cases}
$$

(4) For $\mathcal{H}_1^m(\mu_2,\epsilon) < \mu_1 \le \mathcal{H}_1^{pul}(\mu_2,\epsilon)$ we have that $y_{2m} > 0$.

By the results of Section 3.1.4, these are all true for $m = 1$.

Step 1: We first show that (3.15) is well-defined in the sector Λ^n. For $\mu_1 = \mathcal{H}_1^{pul}(\mu_2,\epsilon)$ we have $y_1 = 0$. This implies the first two equations are for a pulse of o^1 to itself and hence $y_3 = 0$. The next two equations are the first two repeated and so on, hence $y_k = 0$ where k is odd, $1 \le k \le 2n-1$. The even indexed y's are all equal and are less than $y^{*,2}(\epsilon)$ since $0 < y_0 < y^{*,2}(\epsilon)$. This is true for y_{2n} as well. Thus (3.15) is well-defined at $\mu_1 = \mathcal{H}_1^{pul}(\mu_2,\epsilon)$.

For $b^n(\mu_2,\epsilon) \le \mu_1 \le \mathcal{H}_1^{pul}(\mu_2,\epsilon)$, as $\Lambda^n \subset \Lambda^{n-1}$, we have that the first $2n-2$ equations are well-defined. By definition of $b^n(\mu_2,\epsilon)$, so is y_{2n-1}.

Step 2: We now proceed with the shooting. If $\mu_1^+ = \mathcal{H}_1^{pul}(\mu_2,\epsilon)$ then $y_{2n-1}=0$ and hence $y_{2n}^+ = \Pi^{far,1}(0,\mu_1^+) = \mu_1^+ > 0$. On the other hand we show that if $\mu_1^- = b^n(\mu_2,\epsilon)$ then $y_{2n}^- < 0$. As before, we consider two cases. If $\mathcal{H}_1^{n-1}(\mu_2,\epsilon) < b^n(\mu_2,\epsilon)$ then, for $\mu_1^- = b^n(\mu_2,\epsilon)$, we have $(y_{2n-1})^{\lambda_1/\epsilon} = \eta$ and hence, by (3.11) with the index 2 replaced by 1, $y_{2n}^- < 0$. If $\mathcal{H}_1^{n-1}(\mu_2,\epsilon) = b^n(\mu_2,\epsilon)$ then, for $\mu_1^- = \mathcal{H}_1^{n-1}(\mu_2,\epsilon)$, we have $y_{2n-2}=0$. Thus the last two equations of (3.15) are the equations for the pulse of o^2 to itself. As this value of $\boldsymbol{\mu}$ lies above the curve that defines the pulse of o^2, see Remark 3.1.3, we have that $y_{2n}^- < 0$. Hence we can conclude that, by the Intermediate Value Theorem, there exists a value of μ_1, between $\mu_1^- = b^n(\mu_2,\epsilon)$ and $\mu_1^+ = \mathcal{H}_1^{pul}(\mu_2,\epsilon)$, such that $y_{2n}=0$. Choosing the first such value below $\mathcal{H}_1^{pul}(\mu_2,\epsilon)$ defines the function $\mu_1 = \mathcal{H}_1^n(\mu_2,\epsilon)$. The existence proves the first induction hypothesis. Such a curve certainly satisfies $\mathcal{H}_1^{n-1}(\mu_2,\epsilon) \le b^n(\mu_2,\epsilon) < \mathcal{H}_1^n(\mu_2,\epsilon)$, by construction. This is the second induction hypothesis. The third induction hypothesis follows from the construction of $b^n(\mu_2,\epsilon)$. Also, because $\mu_1 = \mathcal{H}_1^n(\mu_2,\epsilon)$ is the first such value below $\mathcal{H}_1^{pul}(\mu_2,\epsilon)$, we have $y_{2n} > 0$ for $\mathcal{H}_1^n(\mu_2,\epsilon) < \mu_1 \le \mathcal{H}_1^{pul}(\mu_2,\epsilon)$. This is the forth induction hypothesis. This completes the induction argument and hence the proof for the 2-dimensional case.

3.2. **The 3-Dimensional Case.**

The proof for the existences of homoclinic pulses and heteroclinic n-fronts in 3-dimensions is very similar to the 2-dimensional case. It is done by a shooting method. The differences lie mainly in the setup, i.e., in the geometry of S^i, Γ^0 and the construction of $\Pi^{far,i}$ and $\Pi^{loc,i}$. We will only prove the existence of a pulse from o^1 to itself. The proof for the n-fronts follow analogously, with modifications that are made apparent in the proof of the pulse.

3.2.1. *Simplification Assumptions.*

Assume there exists a neighborhood of K^i, S^i, such that in this neighborhood S^i, (2.3) has the form

$$\begin{aligned} \dot{x} &= -f(z)x \\ \dot{y} &= g(z)y \\ \dot{z} &= -\epsilon z, \end{aligned} \tag{3.16}$$

where K^i is given by the set $\{(0,0,z) : a \le z \le b\}$, for some $a<0<b$, $o^i=(0,0,0)$, and S^i is given by

$$\{(x,y,z) : |x|,|y| \le \delta, a \le z \le b\}$$

for some $\delta > 0$. Further assume that $f, g > 0$ for $a \le z \le b$. For

$$0 \le \epsilon < \min_{a \le z \le b} \{f(z), g(z)\}$$

the eigenvalues of the linearization of (3.16) at $(0,0,z)$, are

$$-f(z) < -\epsilon < 0 < g(z)$$

and the associated eigenvectors are (1,0,0), (0,1,0), and (0,0,1), respectively. We choose positive constants λ^s and λ^u such that

$$-f(z) \le -\lambda^s < -\epsilon \le 0 \le \epsilon < \lambda^u \le g(z)$$

holds for all $a \le z \le b$. Comparing this to the $\mathbb{R}^2$ model, we notice that we have added another stable direction that is not singular at $\epsilon = 0$. This makes it possible to define the singular heteroclinic loop, see Fig. 4. We note that in general, one can introduce the Fenichel variables ([10, 5, 15, 3]) to cast equation (2.3) in a neighborhood of a normally hyperbolic manifold K in the following form

$$\dot{x} = -F(x,y,z)x, \quad \dot{y} = G(x,y,z)y, \quad \dot{z} = -\epsilon z.$$

Our assumption on Eq. (3.16) simply truncate F, G to their leading order terms: $f(z) = F(0,0,z), g(z) = G(0,0,z)$. Treatment without this linearization assumption can be found in [2], and treatments on other similar situations can be found in [5, 6, 16, 15, 3].

We now describe some known geometric structures in S^i. We note that K^i is clearly invariant, and since $f(z)$ and $g(z)$ are positive, K^i is a normally hyperbolic invariant manifold. We can consider K^i as a center manifold contained in a 2-dimensional center-stable manifold $W^{cs,i}$ given by

$$W^{cs,i} = \{(x,0,z) : a \le z \le b, |x| \le \delta\}.$$

It is also contained in a 2-dimensional center-unstable manifold $W^{cu,i}$ given by

$$W^{us,i} = \{(0,y,z) : a \le z \le b, |y| \le \delta\}.$$

Furthermore, $W^{cs,i}$ and $W^{cu,i}$ are foliated by 1-dimensional invariant stable and unstable foliations, $F^{s,i}(z)$ and $F^{u,i}(z)$, parameterized by points of K^i. These are given by the sets

$$F^{s,i}(z) = \{(x,0,z) : |x| \le \delta\} \text{ and } F^{u,i}(z) = \{(0,y,z) : |y| \le \delta\}.$$

These foliations have the property that if we let $(x,y,z)(t; x_0, y_0, z_0)$ denote the flow defined by (3.16), then the flow through base points of $F^{s,i}(z_0)$ and the flow through the point $(0,0,z_0)$ on K^i satisfy the estimate

$$||(x,y,z)(t; x_0, 0, z_0)) - (0,0,z(t;0,0,z_0))|| = |x(t; x_0, y_0, z_0)| = O(e^{-\lambda^s t}) \tag{3.17}$$

for $t > 0$. For points on $F^{u,i}(z_0)$ we have the estimate

$$||(x,y,z)(t; 0, y_0, z_0) - (0,0,z(t;0,0,z_0))|| = |y(t; 0, y_0, z_0)| = O(e^{-\lambda^u t}) \tag{3.18}$$

for $t < 0$. We point out that such "stratified" geometries for these manifolds and foliations can be generalized without the linearization assumption ([10]). The linearization assumption and the subsequent stratifications simply make our presentation more transparent.

For $0 < \epsilon$, the flow on K^i is given by the last equation of (3.16) and thus o^i is asymptotically stable with respect to the flow on K^i. Hence the flow through points of $F^{s,i}(z)$ converge to o^i as $t \to \infty$.

The singular heteroclinic loop Γ^0 at $\epsilon = 0$ consists of heteroclinic orbits q^i, satisfying

$$\lim_{t\to+\infty} q^i(t) = (0,0,z^{*,j}) \text{ and }$$
$$\lim_{t\to-\infty} q^i(t) = o^i,$$

and pieces of the center manifolds between $z^{*,i}$ and o^i on K^i. Since we assume $z^{*,i} \neq o^i$, then by the choices of local coordinates for S^i we have $z^{*,i} > 0$ (see Fig. 4).

The assumption that $z^{*,i} \neq 0$ guarantees the following. When we perturb this singular loop from $\epsilon = 0$, the resulting heteroclinic orbits will not be degenerate, i.e., they will converge along the leading directions of o^i and o^j. More specifically, when $0 < \epsilon$ we have $q^1(t)$ and $q^2(t)$ persist on the μ_1 and μ_2-axes, respectively, in the following sense. When $\epsilon = 0$ we have $q^i(t)$ lies in the stable manifold of $z^{*,j}$, $W^s((0,0,z^{*,j}))$, which is $F^{s,j}(z^{*,j})$. The points $(0,0,z^{*,j})$ are no longer equilibria for $0 < \epsilon$. We assume that if $\mu_i = 0$ the unstable manifold of o^i, $W^u(o^i)$, lies in the stable foliation $F^{s,j}(z)$ for some z near $z^{*,j}$ and hence the trajectory, denoted by q^i again, converges to o^j as prescribed by (3.17). That means it converges to o^j along the principal stable direction e^j_+ of o^j as $t \to +\infty$.

Finally, unlike the 2-dimensional model, we need to recall that hypothesis **H3** implies what is called the strong inclination property on $T_{q^i(t)}W^{cs,j}$ as $t \to -\infty$ as follows

$$\lim_{t \to -\infty} T_{q^i(t)}W^{cs,j} = T_{o^i}W^u(o^i) + T_{o^i}W^{ss}(o^i),$$

where $T_{o^i}W^{ss}(o^i)$ is the linear space spanned by the eigenvector associated with the eigenvalue $-f(0)$, see Fig. 4. The tangent space on the right is spanned by the tangent vectors $(1, 0, 0)$ and $(0, 1, 0)$.

3.2.2. *Poincaré Maps.* The purpose of this subsection is the same as for the 2-dimensional case, i.e., to construct various flow-induced maps near K^is and between them. We start with Poincaré sections contained in the boundary of S^i

$$\Sigma^{out,i} := \{(x, \delta, z) : |x| \leq \delta,\ |z| \leq \eta_1\} \text{ and } \Sigma^{in,j} := \{(\delta, y, z) : |y| \leq \delta, |z - z^*| \leq \eta_2\},$$

for some constants $0 < \eta_1, \eta_2 \leq \delta$. These sections are transversal to the flow near q^i. The local coordinates in them are simply the local coordinates of S^i and S^j projected onto them. When it is necessary, we will denote the local coordinates of $\Sigma^{in,i}$ by (y^i, z^i) and the local coordinates of $\Sigma^{out,i}$ by (x^i, z^i). Otherwise we will drop the superscripts. Clearly we have the following

$$\begin{aligned}
&\Sigma^{out,i} \cap W^{cu,i} = \{(0, z) : |z| \leq \eta_1\},\\
&\Sigma^{out,i} \cap W^u(o^i) = (0, 0), \text{ and}\\
&\Sigma^{in,i} \cap W^{cs,i} = \{(0, z) : |z| \leq \eta_2\},
\end{aligned}$$

for all $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$ and $0 \leq \epsilon \leq \epsilon_0$, where $\mathbb{R}^2_{loc}$ is a neighborhood of the origin to be specified in more detail later.

Since there are no equilibria in a neighborhood of q^i between $\Sigma^{out,i}$ and $\Sigma^{in,j}$, and since the flow is transversal to these sections, there exist $0 < \epsilon_0$, γ_0 and a flow-induced diffeomorphism $\Pi^{far,i}$ from $\Sigma^{out,i}$ to $\Sigma^{in,j}$ for $\boldsymbol{\mu} \in \mathbb{R}^2_{loc} := \{|\mu_1|,\ |\mu_2| \leq \gamma_0\}$:

$$\Pi^{far,i} : \Sigma^{out,i} \times \mathbb{R}^2_{loc} \times \{0 \leq \epsilon \leq \epsilon_0\} \to \Sigma^{in,j}.$$

It is important to point out that the constants η_1, η_2, ϵ_0, and γ_0 only depend on δ, the size of the neighborhood of K^i. In particular, γ_0 is independent of ϵ.

We now examine the specific form of $\Pi^{far,i}$ and its properties, which we assumed for the 2-dimensional case. We do this by expanding $\Pi^{far,i}$ around the point $(0, 0) = W^u(o^i) \cap \Sigma^{out,i}$. We then neglect the quadratic and higher order terms.

The first term is $\Pi^{far,i}((0, 0), \boldsymbol{\mu}, \epsilon)$ and we write it as

$$\Pi^{far,i}((0, 0), \boldsymbol{\mu}, \epsilon) = (y^j(\boldsymbol{\mu}, \epsilon), z^j(\boldsymbol{\mu}, \epsilon)).$$

Notice that when $\mu_i = 0$ we have $y^j(\boldsymbol{\mu}, \epsilon) = 0$ for all $0 \leq \epsilon \leq \epsilon_0$, because of the existence of the heteroclinic orbit q^i at $\mu_i = 0$. By **H3** and **H4** we can assume without loss of generality that there exists a change of variables in $\mathbb{R}^2_{loc}$ so that $\mu_i = y^j(\boldsymbol{\mu}, \epsilon)$ for all $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$ and $0 \leq \epsilon \leq \epsilon_0$. Notice that $y^j(\boldsymbol{\mu}, \epsilon)$ is simply the Melnikov function defining the separation of $W^u(o^i)$ and $W^{cs,j}$. We also have that $z^j((0, 0), 0) = z^{*,j}$ when $\mu_i = 0$. By truncating the expansion of this function we assume that $z^j(\boldsymbol{\mu}, \epsilon) = z^{*,j}$ for all $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$ and $0 \leq \epsilon \leq \epsilon_0$.

Since $\Pi^{far,i}$ is a diffeomorphism we have that

$$\det(\mathbf{A}((0,0),0)) \neq 0 \text{ where } \mathbf{A}(\boldsymbol{\mu},\epsilon) := D_{(x,z)}\Pi^{far,i}((0,0),\boldsymbol{\mu},\epsilon) = (a_{kl}).$$

Since A is non-singular for all $\boldsymbol{\mu} \in \mathbb{R}^2_{loc}$ and $0 \le \epsilon \le \epsilon_0$ we neglect the high order terms in the expansion of $\mathbf{A}(\boldsymbol{\mu},\epsilon)$ at $(\boldsymbol{\mu},\epsilon) = (0,0)$ and assume that $\mathbf{A} = \mathbf{A}((0,0),0)$. This gives the expansion of $\Pi^{far,i}$ as

$$\Pi^{far,i}((x^i,z^i),\mu_i) = \begin{pmatrix} \mu_i \\ z^{*,j} \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x^i \\ z^i \end{pmatrix} = \begin{pmatrix} y^j \\ z^j \end{pmatrix}.$$

We have in fact obtained that the leading terms of $\Pi^{far,i}$ are independent of μ_j and ϵ, just as what we assumed for the 2-dimensional case.

We now translate the strong inclination property, **H3** and Remark 2, into a property of $\Pi^{far,i}$. Consider the preimage of $W^{cs,j} \cap \Sigma^{in,j} = \{(0,z^j)\}$ under $\Pi^{far,i}$. Solving

$$\Pi^{far,i}((x^i,z^i),\mu_i) = (0,z^j)$$

for (x^i,z^i) yields

$$\begin{pmatrix} x^i \\ z^i \end{pmatrix} = A^{-1} \begin{pmatrix} -\mu_i \\ z^j - z^{*,j} \end{pmatrix} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} -a_{22}\mu_i - a_{12}(z^j - z^{*,j}) \\ a_{21}\mu_i + a_{11}(z^j - z^{*,j}) \end{pmatrix}. \tag{3.19}$$

See Fig. 7. By the strong inclination property, we have that the tangent space of $W^{cs,j}$ at $\mu_i = 0$ under the flow is close to the tangent spaces of $W^u(o^i)$ and $W^{ss}(o^i)$ as $t \to -\infty$. The later spaces are spanned by the vectors $(0,1,0)$ and $(1,0,0)$ in the local coordinates of S^i. In other words, $W^{cs,j} \cap \Sigma^{out,i}$ is approximately the vector $(1,0)$ in the coordinates of $\Sigma^{out,i}$. Thus we have that

$$z^i = \frac{1}{\det(\mathbf{A})}(a_{21}\mu_i + a_{11}(z^j - z^{*,j})) \approx 0$$

when $\mu_i = 0$. So we can conclude $a_{11} \approx 0$, and take $a_{11} = 0$ for simplicity.

We translate the twist assumption, **H5**, into a property of $\Pi^{far,i}$ as well. Recall that $e^i_+ = (0,0,1) \in S^i$ and $e^j_- = (0,1,0) \in S^j$ point to opposite sides of $W^{cs,j}$, see Fig. 4. In the coordinates of $\Sigma^{out,i}$ we have that any vector (x^i,z^i) with $0 < z^i$ points to the same side of $W^{cs,j} \cap \Sigma^{out,i}$ as e^i_+. In the coordinates of $\Sigma^{in,j}$ we have that any vector (y^j,z^j), $y^j < 0$, points to the opposite side of $W^{cs,j} \cap \Sigma^{in,j}$ as e^j_-. Thus, at $\mu_i = 0$, we have

$$\begin{pmatrix} 0 \\ z^{*,j} \end{pmatrix} + \mathbf{A} \begin{pmatrix} 0 \\ z^i \end{pmatrix} = \begin{pmatrix} a_{12}z^i \\ z^{*,j} + a_{22}z^i \end{pmatrix} = \begin{pmatrix} y^j \\ z^j \end{pmatrix}.$$

It follows that $y^j = a_{12}z^i < 0$ when $z^i > 0$, or $a_{12} < 0$.

The strong inclination property together with the twist assumption will give us the subsets of $\Sigma^{out,i}$ on which the y-component of $\Pi^{far,i}$,

$$\mu_i - |a_{12}|z \tag{3.20}$$

is positive or negative. We need to track the sign of the y-coordinate of $\Pi^{far,i}$ because $y > 0$ iff the corresponding point can be further continued without escaping a tubular neighborhood of Γ^0 in which bifurcated pulses and n-fronts are sought after. Recall that $W^{cs,j} \cap \Sigma^{in,j} = \{(0,z)\}$ and hence

$$\mu_i - |a_{12}|z = 0 \text{ if and only if } z = \mu_i/|a_{12}|.$$

This leads to the following characterization:

(1) If $(x^i, z^i) \in \Sigma^{out,i}$ satisfies $0 \le z^i < \mu_i/|a_{12}|$, then the y-component of $\Pi^{far,i}((x^i, z^i), \mu_i)$ is positive;
(2) If $(x^i, z^i) \in \Sigma^{out,i}$ satisfies $\mu_i/|a_{12}| < z^i \le \eta_1$, then the y-component of $\Pi^{far,i}((x^i, z^i), \mu_i)$ is negative.

The condition $\mu_i/|a_{12}| \le \gamma/|a_{12}| < \eta_1$ imposes another constraint on μ_i. We take

$$\gamma < \min\{|a_{12}|\eta_1, \gamma_0\}$$

and reset $\mathbb{R}^2_{loc} = \{|\boldsymbol{\mu}| \le \gamma\}$. This should be compared to (3.11) in the 2-dimensional case. Also, one should compare (3.20) to the form of $\Pi^{far,i}$ in the 2-dimensional case, (3.7). They are exactly the same.

With $\Pi^{out,i}$ we can track solutions near Γ^0 between S^i and S^j. We also wish to track solutions of (2.3) near Γ^0 as they pass through S^i. To do this we define the local map $\Pi^{loc,i}$ from a subset of $\Sigma^{in,i}$ to $\Sigma^{out,i}$. This map will be singular at $\epsilon = 0$, as in the 2-dimensional case.

To begin we consider the so-called Shilnikov solutions of (3.16) ([6, 7, 17, 3]). Let $\mathcal{A}(t) := \int_0^t f(z(s))ds$ and $\mathcal{B}(t,\tau) := \int_\tau^t g(z(s))ds$, $0 \le t \le \tau$, $0 < \tau$, and consider the functions

$$\begin{aligned} x(t;\tau,z_0) &:= \delta e^{-\mathcal{A}(t)} \\ y(t;\tau,z_0) &:= \delta e^{\mathcal{B}(t,\tau)} \\ z(t;\tau,z_0) &:= z_0 e^{-\epsilon t}, \end{aligned} \tag{3.21}$$

with $|z_0 - z^{*,i}| \le \eta_2$. Since $0 \le z(t; z_0, \tau) \le z_0$, and as $f, g > 0$, and $0 < t \le \tau$ we have that $0 < x(\tau; z_0, \tau) < \delta$ and $0 < y(0; z_0, \tau) < \delta$ as $B(t, \tau) < 0$. It is straightforward to check that (3.21) defines a solution to (3.16). Furthermore, if

$$\tau \ge \ln((z_0/\eta_1)^{1/\epsilon}) := \tau^*(z_0, \epsilon),$$

then $0 < z(\tau; z_0, \tau) \le \eta_1$. In other words, $\tau^*(z_0, \epsilon)$ measures the time required for the solution to cross $z = \eta_1$ from above from a neighborhood $|z_0 - z^{*,i}| \le \eta_2$ of $z^{*,i}$ which lies above $z = \eta_1$ because of the choice of the coordinates of S^i and the small magnitude of η_1, η_2. Examining $\tau^*(z_0, \epsilon) = \ln((z_0/\eta_1)^{1/\epsilon})$ we see that if $\eta_1 < z_0$ then as $\epsilon \to 0^+$, $\tau^*(z_0, \epsilon) \to \infty$. Since both $x(\tau; z_0, \tau)$ and $y(0; z_0, \tau)$ are decreasing in τ, we may choose $\epsilon_0 > 0$ such that if $0 \le \epsilon \le \epsilon_0$ and $\tau \ge \tau^*(z_0, \epsilon)$, then

$$x(\tau; \tau, z_0) = \delta e^{-\mathcal{A}(\tau)} \le \delta,\ y(0; z_0, \tau) = \delta e^{\mathcal{B}(0,\tau)} \le \delta,$$

uniformly in $|z_0 - z^{*,i}| \le \eta_2$. It then follows that the initial point of the solution (3.21) is in $\Sigma^{in,i}$ because

$$(x(0; \tau, z_0), y(0; \tau, z_0), z(0; \tau, z_0)) = (\delta, y(0; \tau, z_0,), z_0) \in \Sigma^{in,i}$$

and the corresponding end point at $t = \tau$ of (3.21) is in $\Sigma^{out,i}$ because

$$(x(\tau; \tau, z_0), y(\tau; \tau, z_0), z(\tau; \tau, z_0)) = (x(\tau; \tau, z_0), \delta, z(\tau; \tau, z_0)) \in \Sigma^{out,i}.$$

Therefore, the flow (3.21) with the choices in $\epsilon_0, \tau^*(z_0, \epsilon)$ above defines a map, denoted by $\Pi^{loc,i}$ from $\Sigma^{in,i}$ to $\Sigma^{out,i}$. Note that this map is parameterized by the initial value in z, i.e., z_0, and the time of flight, $\tau \ge \tau^*(z_0, \epsilon)$, of the flow from $\Sigma^{in,i}$ to $\Sigma^{out,i}$. See Fig.7 for a 3-dimensional illustration of the local flow that defines $\Pi^{loc,i}$.

Our strategy next is to write the time of flight τ as a function of (y_0, z_0), with $y_0 = y(t; \tau, z_0)$ being the initial value in y. First the domain and range of $\Pi^{loc,i}$ are

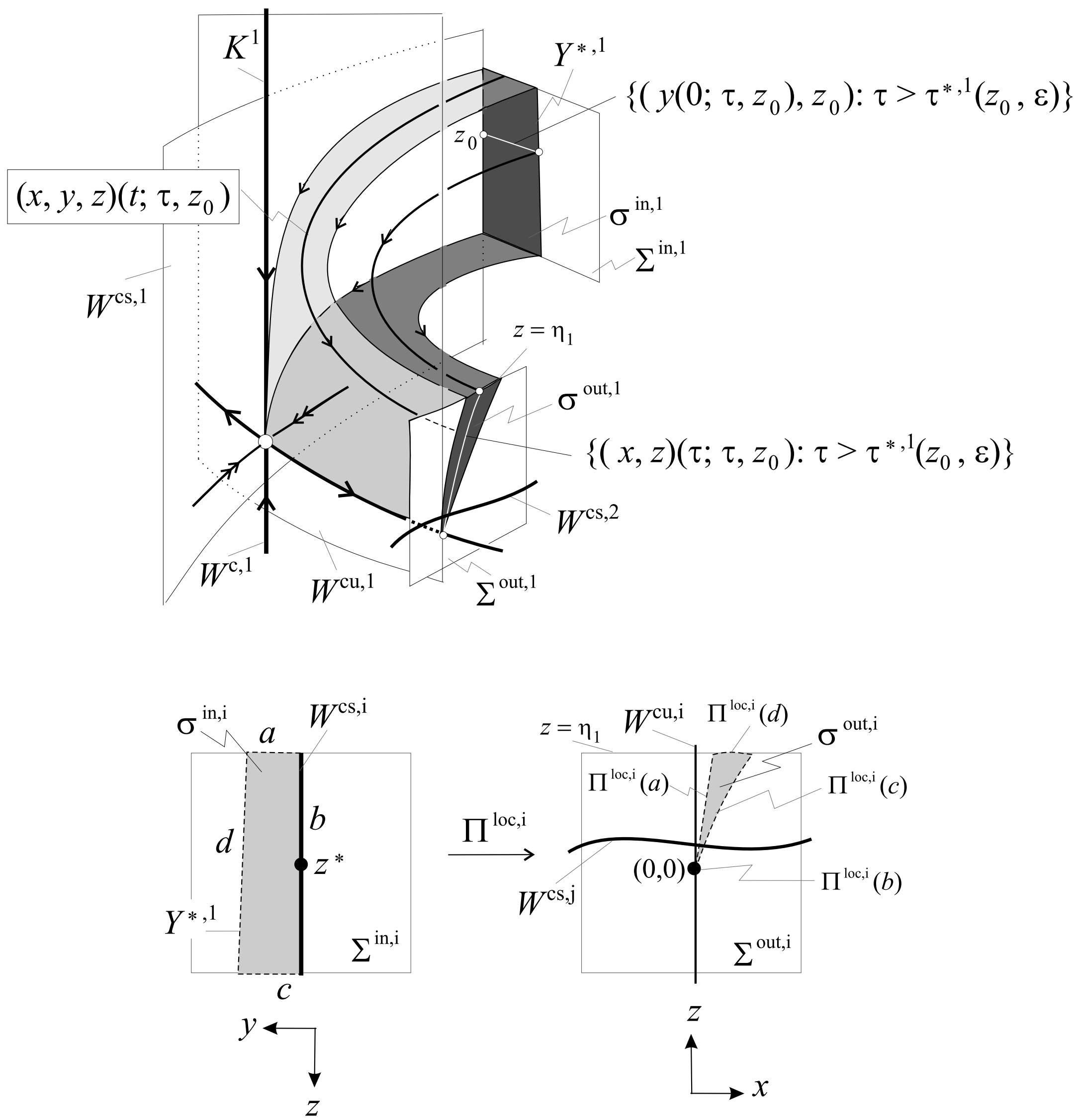

FIGURE 7. *Top*: Local flows near K^1 in 3-dimension that define the local map $\Pi^{loc,1}$. *Bottom*: More details for the 2-dimensional local map $\Pi^{loc,i}$.

given as follows

$$\sigma^{in,i} := \{(y(0;\tau,z_0),z_0) : |z_0 - z^{*,i}| \leq \eta_2,\ \tau \geq \tau^*(z_0,\epsilon)\} \subset \Sigma^{in,i}$$

and

$$\sigma^{out,i} := \{(x(\tau;\tau,z_0),z(\tau;\tau,z_0)) : |z_0 - z^{*,i}| \leq \eta_2,\ \tau \geq \tau^*(z_0,\epsilon)\} \subset \Sigma^{out,i}.$$

For reference purposes, we let

$$Y^{*,i}(z_0, \epsilon) := y(0; \tau^*(z_0, \epsilon), z_0),$$

See Fig.7 for an illustration. We point out that $\sigma^{in,i}$ does not contain the set $\{(0, z_0)\} = W^{cs,i} \cap \Sigma^{in,i}$ since solutions of (3.16) on $W^{cs,i}$ converge to o^i as $\tau \to \infty$ (for $\epsilon > 0$) and o^i is not in $\Sigma^{out,i}$. However, it is on the boundary since $0 < y(0; z_0, \tau) \to 0$ as $\tau \to \infty$. Also, $\sigma^{out,i}$ has a cusp shape tangent to the positive z-axis of $\Sigma^{out,i}$ with the point of the cusp at $(0, 0)$. To see this examine the slope of the individual curves in the union defining $\sigma^{out,i}$,

$$\frac{x(\tau; \tau, z_0)}{z(\tau; \tau, z_0)} = \frac{\delta e^{-\mathcal{A}(\tau)}}{z_0 e^{-\epsilon\tau}} = \frac{\delta}{z_0} e^{-\mathcal{A}(\tau)+\epsilon\tau} = O(e^{(-\lambda+\epsilon)\tau})$$

as $\tau \to \infty$, see Fig. 7.

To write τ as a function of (y_0, z_0) we consider the set

$$\Delta^i := \{(\tau, z_0) : |z_0 - z^{*,i}| \leq \eta_2,\ \tau^*(z_0, \epsilon) \leq \tau\}.$$

Clearly there exist two smooth functions, ρ^u and ρ^s, mapping Δ^i into $\sigma^{out,i}$ and $\sigma^{in,i}$, respectively, by

$$\begin{array}{lll} (\tau, z_0) \in \Delta^i & \stackrel{\rho^u}{\rightarrow} & (x(\tau; \tau, z_0), z(\tau; \tau, z_0)) \in \sigma^{out,i} \\ & \stackrel{\rho^s}{\searrow} & \\ & & (y(0; \tau, z_0,), z_0) \in \sigma^{in,i}. \end{array}$$

These maps are bijective by the construction. Examining the determinant of the derivative of ρ^s yields

$$\left| \begin{array}{cc} \frac{\partial y}{\partial \tau} & \frac{\partial y}{\partial z_0}(0; \tau, z_0) \\ 0 & 1 \end{array} \right| = \frac{\partial y}{\partial \tau}(0; \tau, z_0) = -e^{\mathcal{B}(0,\tau)} g(z(\tau; \tau, z_0)) > 0,$$

and thus, by the Inverse Function Theorem, ρ^s is a diffeomorphism. Implicitly this defines τ as a function of (y_0, z_0). Hence $\Pi^{loc,i}$ from $\sigma^{in,i}$ onto $\sigma^{out,i}$ can be written as

$$\Pi^{loc,i}((y_0, z_0), \epsilon) = \rho^u \circ (\rho^s)^{-1}((y_0, z_0), \epsilon).$$

We extend $\Pi^{loc,i}$ onto the closure of $\sigma^{in,i}$ (that is we include $W^{cs,i} \cap \Sigma^{in,i}$ in the definition of $\sigma^{in,i}$) by the fact that $\tau \to \infty$ if and only if

$$x(\tau; \tau, z_0), y(0; \tau, z_0), z(\tau; \tau, z_0) \to 0.$$

Hence we define

$$\Pi^{loc,i}((0, z_0), \epsilon) = (0, 0).$$

Finally, when writing $\Pi^{loc,i}$ in component form we will use the notation

$$\Pi^{loc,i} = (X^i(y_0, z_0, \epsilon), Z^i(y_0, z_0, \epsilon)).$$

Remark 3.3: For each z_0 fixed, the τ-parameterized curve in $\sigma^{out,i}$ crosses $W^{cs,j} \cap \Sigma^{out,i}$ transversely exactly once if at all (see Fig. 7). To see this we note that $W^{cs,j} \cap \Sigma^{out,i}$, given in (3.19), is a curve transversal to the z-axis of $\Sigma^{out,i}$, more or less parallel to the x-axis, and for $\mu_i > 0$ this intersection lies above $(0, 0)$. The cusp of $\sigma^{out,i}$ is tangent to the z-axis at $(0, 0)$, and as $\epsilon \to 0$ they become more so (steeper). This implies that the crossing is transversal. The crossing is unique because $z(\tau; z_0, \tau)$ increase monotonically to η_1 as y_0 increase to $Y^{*,i}(z_0, \epsilon)$, or equivalently $\tau(y_0, z_0)$ decreases to $\tau^*(z_0, \epsilon)$. This fact will allow us to uniquely define $\mathcal{H}_1^{pul}$. $\Diamond$

3.2.3. *Homoclinic Bifurcation.* Recall the definition of a pulse of o^1 around o^2 made in the previous section. It remains the same for the 3-dimensional case except that its passing through S^2 requires the trajectory to spend a time of $O(1/\epsilon)$-order near K^i. Our goal is to show the existence of a pulse bifurcation curve in $\mathbb{R}^2_{loc}$, $\mu_1 = \mathcal{H}_1^{pul}(\mu_2, \epsilon)$, whose domain of definition is independent of ϵ.

Finding a pulse of o^1 is equivalent to finding

$$\boldsymbol{\mu} \in \mathbb{R}^2_{loc}, \text{ with } y_1 = 0$$

to satisfy the following equations

$$\begin{array}{lcl} \Pi^{far,1}(0,0,\boldsymbol{\mu}) & = & \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \\ \Pi^{far,2}(X^2(y_0,z_0,\epsilon), Z^2(y_0,z_0,\epsilon)) & = & \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ z_1 \end{pmatrix}. \end{array} \tag{3.22}$$

We will use a shooting method in two steps as we did in the 2-dimensional case.
Step 1: We need to show that the compositions implied by these equations are well-defined. Notice that

$$\Pi^{far,1}(0,0,\boldsymbol{\mu}) = \begin{pmatrix} \mu_1 \\ z^{*,2} \end{pmatrix} = \begin{pmatrix} y_0 \\ z_0 \end{pmatrix},$$

thus we need $0 \leq y_0 = \mu_1 < Y^{*,2}(z^{*,2}, \epsilon) := y^{*,2}(\epsilon)$, the same relation as illustrated in Fig. 5.
Step 2: We now perform the shooting. We need to show that for each fixed $0 < \epsilon \leq \epsilon_0$ and $0 < \mu_2 \leq \gamma$, if $\mu_1^+ = 0$ then $y_1^+ > 0$, and if $\mu_1^- = y^{*,2}(\epsilon)$ then $y_1^- < 0$. Indeed, setting $\mu_1^+ = y_0 = 0$ implies that $(X^2(y_0,z_0,\epsilon), Z^2(y_0,z_0,\epsilon)) = (0,0)$ and hence, by the characterization of $\Sigma^{out,2}$ given at the end of the section of the global map, we have $y_1^+ = \mu_2 > 0$. On the other hand, we show that if we let

$$\mu_1^- = y_0 = y^{*,2}(\epsilon)$$

then $y_1^- < 0$. Indeed, by the definition of $y^{*,2}(\epsilon)$ we have

$$Z^2(y_0, z_0) = \eta_1 > \gamma/|a_{12}| \geq \mu_2/|a_{12}|,$$

thus, by the characterization of $\Sigma^{out,2}$ given at the end of the section of the global map, $y_1^- < 0$. As everything is continuous, by the Intermediate Value Theorem, there exists a value of μ_1, between $\mu_1^+ = 0$ and $\mu_1^- = y^{*,2}(\epsilon)$, such that $y_1 = 0$. This defines the function $\mu_1 = \mathcal{H}_1^{pul}(\mu_2, \epsilon) \in (0, y^{*,2}(\epsilon)), \mu_2 \in (0, \gamma]$. This function is unique since the curves in $\sigma^{out,i}$ defined by moving $y_0 = \mu_1$ from 0 to $y^{*,2}(\epsilon)$ crosses $W^{cs,1} \cap \sigma^{out,2}$ exactly once and transversely, as stated in Remark 3.2.2. Finally, we point out that $0 < \mu_2 \leq \gamma$ is fixed, independent of ϵ, thus the domain of definition of $\mathcal{H}_1^{pul}$ is independent of ϵ.

The proof for n-fronts follows much the same as the proof in the 2-dimensional case. This concludes the proof for the 3-dimensional case.

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