Weak Lefschetz Property for Ideals Generated by Powers of Linear Forms

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Research Question

Do ideals generated by powers of linear forms in $\mathbb{P}^2$ exhibit the **Weak Lefschetz Property?**

Weak Lefschetz Property (WLP)

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_r]$ be an ideal such that $A = S/I$ is Artinian. A possesses the **Weak Lefschetz Property (WLP)** if there is a linear form ℓ such that, for all m, the map $A_m \xrightarrow{\cdot \ell} A_{m+1}$ is either injective or surjective. If such ℓ exists, then the generic linear form will also have this property.

Our Case

We study the **WLP** for ideals generated by powers of linear forms. This fits into the framework of the following known results:

- **Anick** shows in [1] that if I is generated by generic forms, then A has WLP. We do not require the genericity assumption.
- **Brenner and Kaid** [2] show that if the syzygy bundle of an almost complete intersection in $\mathbb{P}^2$ is not semistable, then A has WLP. Our theorem applies both to semistable and non-semistable syzygy bundles.
- **Migliore, Miró-Roig, Nagel** [4] focus on the case of **WLP** for monomial ideals, showing that WLP can change if even in one of the generators a a factor is replaced by a linear form. Our approach can be considered a further investigative step in this direction.

References

[1] D. Anick, Thin algebras of embedding dimension three, *J. Algebra*, **100** (1986), 235–259.

[2] H. Brenner, A. Kaid, Syzygy bundles on $\mathbb{P}^2$ and the weak Lefschetz property. *Illinois J. Math.* **51** (2007), 1299–1308.

[3] A. Geramita, H. Schenck, Fatpoints, inverse systems, and piecewise polynomial functions, *J. Algebra*, **204** (1998), 116–128.

[4] J. Migliore, U. Nagel, R. Miró-Roig, Monomial ideals, almost complete intersections and the weak Lefschetz property preprint, 2009.

[5] J. Migliore, R. Miró-Roig, Ideals of general forms and the ubiquity of the weak Lefschetz property. *J. Pure Appl. Algebra* **182** (2003), 79–107.

Main Result and Method

An Artinian quotient of $\mathbb{K}[x, y, z]$ by powers of linear forms has WLP.

- The long exact sequence in cohomology of the restriction of the syzygy bundle to a line L yields:

$$0 \longrightarrow H^0(Syz(I)(m)) \longrightarrow H^0(Syz(I)(m+1)) \xrightarrow{\phi_m} H^0(Syz(I)|_L(m+1))$$

$$\longrightarrow H^1(Syz(I)(m)) \xrightarrow{\cdot \ell} H^1(Syz(I)(m+1)) \longrightarrow H^1(Syz(I)|_L(m+1))$$

$$\xrightarrow{\psi_m} H^2(Syz(I)(m)) \longrightarrow H^2(Syz(I)(m+1)) \longrightarrow H^2(Syz(I)|_L(m+1)) = 0.$$

Therefore injectivity of $A_m \to A_{m+1}$ follows from surjectivity of ϕ_m and surjectivity from injectivity of ψ_m.

- From the knowledge of resolutions of ideals generated by linear forms (see below), we deduce

$$Syz(I) \otimes S/\ell \simeq S/\ell(-\omega-2)^a \oplus S/\ell(-\omega-1)^{n-1-a}$$

- Suppose $m < \omega$. Then multiplication by ℓ is injective since the source of ϕ_m is zero

$$H^0(Syz(I)|_L(m+1) \simeq H^0(\mathcal{O}_L(m-1-\omega))^a \oplus H^0(\mathcal{O}_L(m-\omega)))^{n-1-a} = 0$$

- If instead $m \geq \omega$, multiplication by ℓ is surjective since, by Serre duality, the target of ψ_m is zero

$$H^1(Syz(I)|_L(m+1) \simeq H^0(\mathcal{O}_L(-m-1+\omega))^a \oplus H^0(\mathcal{O}_L(-m-2+\omega)))^{n-1-a} = 0$$

Weak Lefschetz & Syzygy Bundle

- **Definition** *If $I = \langle f_1, \ldots, f_n \rangle$ with $deg(f)_i = d_i$ and I is $m = \langle x_1, \ldots, x_r \rangle$ primary, then the* ***syzygy bundle*** *Syz is a rank $n-1$ bundle defined via:*

$$0 \longrightarrow Syz(I) \longrightarrow \bigoplus_{i=1}^{n} S(-d_i) \longrightarrow S.$$

At the level of modules the cokernel of the rightmost map is S/I, but upon sheafifying it vanishes.

$$0 \to Syz(I)(m) \to \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^2}(m-d_i) \to \mathcal{O}_{\mathbb{P}^2}(m) \to 0.$$

- **Brenner and Kaid** [2] show that if $A = S/I$ then

$$A = \bigoplus_{m \in \mathbb{Z}} H^1(Syz(I)(m)).$$

- For semistable ideals in in $\mathbb{P}^2$, they give sufficient and necessary conditions for WLP in terms of the generic splitting type of the syzygy bundle.

Resolutions of Linear Forms Ideals

Proposition 1 ***(Geramita and Schenck, [3])*** *Given an ideal of $\mathbb{K}[y, z]$ with minimal generating set $J = \langle l_1^{\alpha_1}, \ldots, l_t^{\alpha_t} \rangle$, with l_i linear forms, J has minimal free resolution*

$$0 \to S(-\omega-2)^a \oplus S(-\omega-1)^{t-1-a} \to \bigoplus_{i=1}^{t} S(-\alpha_i) \to J.$$

where

- $\omega = \left\lfloor \frac{\sum_{i=1}^t \alpha_i - t}{t-1} \right\rfloor$ *is the socle degree of* $\mathbb{K}[y, z]/J$,
- $a = \sum_{i=1}^t \alpha_i - (t-1)(\omega - 1)$.

The resolution does not depend on the linear forms, but only on the arithmetic of the powers.

Why this result is best possible

- **WLP need not hold for ideals generated by powers of linear forms in more than three variables.**

Example 1 The ring

$$A = \mathbb{K}[x, y, z, w]/\langle x^3, y^3, z^3, w^3, (x+y)^2, (z+w)^2 \rangle$$

does not have **WLP**. The Hilbert function of A is $(1, 4, 8, 8, 4)$ and a computation shows that the map $A_2 \longrightarrow A_3$ does not have full rank.

- **Although the syzygy bundles studied by us are tipycally not semistable, this hypothesis is not sufficient for WLP to hold outside the realm of complete intersections.**

Example 2 The ring

$$A = \mathbb{K}[x, y, z]/\langle x^5, y^5, z^5, x^2yz, xy^2z \rangle$$

does not have **WLP** although the syzygy bundle associated to the monomial ideal is not semistable.

Open Questions

- As shown by authors' computations, the non-WLP locus seems to have a combinatorial description in terms of the hyperplane arrangement given by the linear forms.
  - **Give a combinatorial characterization of the non-WLP locus. Discuss connections between WLP and hyperplane arrangements theory.**
- As shown in **Migliore, Miró-Roig, Nagel** [4], WLP behaves in very subtle ways in positive characteristic.
  - **What about ideals generated by powers of linear forms in positive charactersitic?**
- As shown by Example 1 the result does not hold in more than three variables. We ask:
  - **What further conditions are needed to have WLP for ideals generatd by powers of linear forms in higher number of variables?**

Acknowledgements

Computations were performed using Macaulay2, available at: `http://www.math.uiuc.edu/Macaulay2/` and scripts to analyze WLP are available on the second author's webpage: `http://www.math.uiuc.edu/~asecele2`