Weak Lefschetz Property for Ideals Generated by Powers of Linear Forms

Alexandra Seceleanu

University of Illinois at Urbana-Champaign

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Weak Lefschetz Property

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_r]$ be an ideal such that the graded algebra $A = S/I$ is Artinian.

Definition

A has the **Weak Lefschetz Property** *(WLP) if there is a linear form ℓ such that, for all m, the map $A_m \xrightarrow{\cdot \ell} A_{m+1}$ is either injective or surjective (i.e. has maximal rank as a vector space map).*

If such ℓ exists, then a generic linear form will also have this property.

Our Theorem

Theorem (Schenck,_)

An Artinian quotient of $\mathbb{K}[x, y, z]$ by an ideal generated by powers of linear forms has the Weak Lefschetz Property.

Motivation

For $A = \mathbb{K}[x, y, z]/I$

- **Anick** shows that if I is generated by generic forms, then A has WLP. We do not require the genericity assumption.
- **Brenner and Kaid** show that if the syzygy bundle of an almost complete intersection in $\mathbb{P}^2$ is not semistable, then A has WLP. Our theorem applies both to semistable and non-semistable syzygy bundles.
- **Migliore, Miró-Roig, Nagel** discuss an instance where WLP changes upon replacing a variable by a linear form. Our result points out a class of ideals whose WLP behaviour is preserved by linear transformations.

Syzygy Bundle

Definition

If $I = \langle f_1, \ldots, f_n \rangle$ *with* $deg(f_i) = d_i$, *then the* **syzygy bundle** Syz *is a rank* $n-1$ *bundle defined via:*

$$0 \to Syz(I)(m) \to \bigoplus_{i=1}^{n-1} \mathcal{O}_{\mathbb{P}^2}(m - d_i) \to \mathcal{O}_{\mathbb{P}^2}(m) \to 0.$$

If $A = S/I$ then $A = \bigoplus_{m \in \mathbb{Z}} H^1(Syz(I)(m))$.

The Syzygy Bundle Technique

Harima-Migliore-Nagel-Watanabe introduced the syzygy bundle of I to study the WLP.

The long exact sequence in cohomology given by the restriction of the syzygy bundle to a line L yields:

$$0 \to H^0(Syz(I)(m)) \to H^0(Syz(I)(m+1)) \xrightarrow{\phi_m} H^0(Syz(I)|_L(m+1)) \to$$
$$H^1(Syz(I)(m)) \xrightarrow[\cdot \ell]{} H^1(Syz(I)(m+1)) \xrightarrow[\psi_m]{} H^1(Syz(I)|_L(m+1)) \to$$
$$H^2(Syz(I)(m)) \longrightarrow \cdots .$$

Syzygies of Powers of Linear Forms

Theorem (Geramita-Schenck)

An ideal $I = \langle l_1^{\alpha_1}, \ldots, l_t^{\alpha_t} \rangle \subseteq \mathbb{K}[y,z]$ *minimally generated by powers of the linear forms* l_i *has free resolution*

$$0 \to S(-\omega-2)^a \oplus S(-\omega-1)^{t-1-a} \to \bigoplus_{i=1}^{t} S(-\alpha_i) \to I,$$

where $\omega = \left\lfloor \frac{\sum_{i=1}^t \alpha_i - t}{t-1} \right\rfloor$ *and* $a = \sum_{i=1}^t \alpha_i - (t-1)(\omega-1)$.

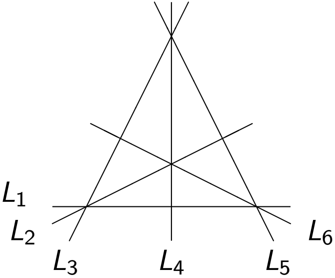
This allows us to compute

$$Syz(I)|_L \simeq \mathcal{O}|_L(-\omega-2)^a \oplus \mathcal{O}|_L(-\omega-1)^{t-1-a} \oplus \mathcal{O}|_L(-d_i).$$

where the sum in red is over nonminimal generators.

Example

Let $I = \langle x^5, (x-y)^2, y^5, (y-z)^2, z^5, (x-z)^2 \rangle$ be the ideal corresponding to the configuration of lines depicted below:



The restriction of I to a generic line has the splitting

$$Syz(I)|_L \simeq \mathcal{O}|_L(-3)^2 \oplus \mathcal{O}|_L(-5)^3.$$

Idea of Our Proof

- Suppose $m < \omega$. Then multiplication by ℓ is injective since the source of ϕ_m is zero

$$H^0 Syz(I)|_L(m+1) \simeq H^0\mathcal{O}_L(m-1-\omega)^a \oplus H^0\mathcal{O}_L(m-\omega)^{n-1-a} = 0$$

- If instead $m \geq \omega$, multiplication by ℓ is surjective since, by Serre duality, the target of ψ_m is zero

$$H^1 Syz(I)|_L(m+1) \simeq H^0\mathcal{O}_L(-m-1+\omega)^a \oplus H^0\mathcal{O}_L(-m-2+\omega)^{n-1-a}$$

$$= 0.$$

Why this Result is Best Possible

- WLP need not hold for ideals generated by powers of linear forms in four or more variables.

$$A = \mathbb{K}[x, y, z, w]/\langle x^3, y^3, z^3, w^3, (x+y)^2, (z+w)^2\rangle$$

  does not have WLP.

- What further (combinatorial) conditions are needed to have WLP for ideals generated by powers of linear forms in higher number of variables?