Inverse systems, fat points and
the Weak Lefschetz Property

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The Weak Lefschetz Property

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_r]$ be an ideal such that $A = S/I$ is Artinian.

**Definition**

A graded Artinian algebra $A$ has the **Weak Lefschetz Property (WLP)** if there is an element $\ell \in S_1$ such that for all degrees $j$, the map $\mu_\ell : A_j \rightarrow A_{j+1}$ is either injective or surjective (equivalently $\mu_\ell$ has maximum rank as a $\mathbb{K}$-vector space homomorphism).
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- The set of linear forms $\ell$ with this property is Zariski open in $S_1$.
- We assume henceforth that $\ell \in S_1$ is generic.
- We assume also that $char(\mathbb{K}) = 0$. 
WLP is

- known to hold for monomial Artinian complete intersections (Stanley, 1980).
- expected to hold for Artinian ideals generated by generic forms.
- known to hold for Artinian ideals generated by generic forms in $\mathbb{K}[x_1, x_2, x_3]$ (Anick, 1986).
- known to hold for any Artinian ideal generated by powers of linear forms in $\mathbb{K}[x_1, x_2, x_3]$ (Schenck - S., 2009).
- known to hold for Artinian ideals generated by generic forms in $\mathbb{K}[x_1, x_2, x_3, x_4]$ (Migliore - Miró-Roig, 2003).
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- known to hold for Artinian ideals generated by *generic* forms in $\mathbb{K}[x_1, x_2, x_3]$ (Anick, 1986).
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How about powers of linear forms in $\mathbb{K}[x_1, x_2, x_3, x_4]$?
Motivating Example: 5 points on a conic

Look ahead:

\[ I = (\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3) \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]. \]
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- The space of quartics in \( \mathbb{P}^2 \) passing through five double points is nonempty \( \implies \) WLP fails for geometric reasons.
Let \( \{p_1, \ldots, p_n\} \subseteq \mathbb{P}^{r-1} \) be a set of distinct points defined by ideals \( I(p_i) = \mathfrak{p}_i \subseteq R = \mathbb{K}[y_1, \ldots, y_r] \). A **fat point ideal** is an ideal of the form

\[
F = \bigcap_{i=1}^{n} \mathfrak{p}_i^{m_i} \subseteq R.
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Recall \( S = \mathbb{K}[x_1, \ldots, x_r] \) and define an action of \( R \) on \( S \) by partial differentiation: 
\[
y_j \cdot x_i = \frac{\partial x_i}{\partial x_j}.
\]

**Definition**

The set of elements annihilated by the action of \( F \) is denoted \( F^{-1} \) and called the (Macaulay) inverse system associated to the ideal \( F \).
Linear forms come into play

Emsalem and Iarrobino proved that there is a close connection between ideals generated by powers of linear forms and ideals of fat points.

**Theorem (Emsalem and Iarrobino)**

Let $F$ be an ideal of fatpoints:

$$F = \mathcal{P}_1^{m_1} \cap \cdots \cap \mathcal{P}_n^{m_n} \subset R.$$
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**Theorem (Emsalem and Iarrobino)**

Let $F$ be an ideal of fat points:

$$F = \varphi_1^{m_1} \cap \cdots \cap \varphi_n^{m_n} \subset R.$$ 

Then

$$(F^{-1})_j = \begin{cases} S_j & \text{for } j < \max\{m_i\} \\ L_{p_1}^{j-m_1+1} S_{m_1-1} + \cdots + L_{p_n}^{j-m_n+1} S_{m_n-1} & \text{for } j \geq \max\{m_i\} \end{cases}$$

and

$$\dim_K(F^{-1})_j = \dim_K(R/F)_j.$$
Second tool - the syzygy bundle

Harima-Migliore-Nagel-Watanabe have introduced the syzygy bundle as a crucial tool in studying the WLP.

**Definition**

If \( I = \langle f_1, \ldots, f_n \rangle \) is \( \langle x_1, \ldots, x_r \rangle \)-primary, and \( \deg(f_i) = d_i \), then the **syzygy bundle** \( S(I) = \tilde{\text{Syz}}(I) \) is a rank \( n - 1 \) bundle defined via

\[
0 \rightarrow \text{Syz}(I) \xrightarrow{} \bigoplus_{i=1}^{n} S(-d_i)[f_1, \ldots, f_n] \xrightarrow{} S \rightarrow S/I \rightarrow 0.
\]

Most importantly, \( H^1(S(I)(j)) = A_j \).
The long exact sequence in cohomology given by the restriction of the syzygy bundle to a hyperplane $L$ defined by the linear form $l$ yields:

$$
0 \rightarrow H^0(S(I)(j)) \rightarrow H^0(S(I)(j + 1)) \rightarrow H^0(S(I)|_L(j + 1))
$$

$$
\rightarrow A_j \xrightarrow{\cdot l} A_{j+1} \rightarrow H^1(S(I)|_L(j + 1))
$$

$$
\rightarrow H^2(S(I)(j)) \rightarrow \cdots
$$
Consider the fat points ideal \( F = \mathcal{O}_{1}^{m_1} \cap \cdots \cap \mathcal{O}_{n}^{m_n} \subset R \).

On the blowup \( X \) of \( \mathbb{P}^{r-1} \) at the points \( p_1, \ldots, p_n \), let

- \( E_i \) be the class of the exceptional divisor over the point \( p_i \)
- \( E_0 \) be the pullback of a hyperplane on \( \mathbb{P}^{r-1} \)

The divisor

\[
D_j = jE_0 - \sum_{i=1}^{n} (j - m_i + 1)E_i.
\]

describes the global sections of the syzygy bundle

\[
h^0(\mathcal{S}(I)(j)) = h^1(D_j)
\]
**Definition**

A linear system of degree $d$ through a set of fat points $\wp_1, \ldots, \wp_n$ with multiplicities $m_1, \ldots, m_n$ in $\mathbb{P}^2$ is **special** if its dimension exceeds the expected dimension $(\frac{d+2}{2}) - \sum_{i=1}^{n} \left( \frac{m_i+1}{2} \right) - 1$.

- E.g. The linear system of quartics through 5 double points has negative expected dimension, but its actual dimension is 1.

- By Riemann-Roch, the space of global sections (or $H^0$ cohomology) is larger than expected iff the $H^1$ cohomology $\neq 0$.

**Definition**

We say $D = dE_0 - \sum_{i=1}^{n} m_iE_i$ is **special** if $h^0(D)$ and $h^1(D)$ are positive.
Motivating Example revisited

Let \( I = (\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3) \subset S = \mathbb{K}[x_1, x_2, x_3, x_4] \) and let \( A = S/I \).

The Hilbert function of \( A \) is:

\[
\begin{array}{cccccccc}
  j & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\
\dim_{\mathbb{K}} A_j & 1 & 4 & 10 & 15 & 15 & 6 & 0 & \ldots \\
\end{array}
\]
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\[
0 \rightarrow H^0(S(I)(3)) \rightarrow H^0(S(I)(4)) \rightarrow H^0(S(I)|_L(4)) \rightarrow A_3 \xrightarrow{\cdot \ell} A_4 \rightarrow H^1(S(I)|_L(4)) \rightarrow H^2(S(I)(3)) \rightarrow \cdots
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\[0 \longrightarrow H^0(S(I)(3)) \longrightarrow H^0(S(I)(4)) \longrightarrow H^0(S(I)|_L(4)) \longrightarrow \]

\[H^0(S(I)(3)) \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[A_3 \quad \ell \quad A_4 \]

\[H^0(S(I)|_L(4)) \quad H^1(S(I)|_L(4)) \]

\[H^1(S(I)(3)) \longrightarrow \ldots \]
Special divisors and (-1)-curves in $\mathbb{P}^2$

**Conjecture (Segre-Harbourne-Gimigliano-Hirschowitz SHGH)**

If $D = dE_0 - \sum_{i=1}^{n} m_i E_i$ is a special divisor on a blowup of $\mathbb{P}^2$, then there exists a $(-1)$-curve $E$ with $E \cdot D \leq -2$. ($(-1)$-curve means $E \cdot E = -1$)

- This conjecture is known to be true for $n \leq 8$ points.
Special divisors and \((-1)\)-curves in \(\mathbb{P}^2\)

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**Theorem (S.)**

If \(E = dE_0 - \sum_{i=1}^{8} m_i E_i\) is the divisor of a \((-1)\)-curve on a blowup of \(\mathbb{P}^2\) at \(n \leq 8\) points, then the coefficients are given by

- \(d = 0, m_i = (-1, 0, 0, 0, 0, 0, 0, 0)\)
- \(d = 1, m_i = (0, 0, 0, 0, 0, 0, 1, 1)\)
- \(d = 2, m_i = (0, 0, 0, 1, 1, 1, 1, 1)\)
- \(d = 3, m_i = (0, 1, 1, 1, 1, 1, 1, 2)\)
- \(d = 4, m_i = (1, 1, 1, 1, 1, 2, 2, 2)\)
- \(d = 5, m_i = (1, 1, 2, 2, 2, 2, 2, 2)\)
- \(d = 6, m_i = (2, 2, 2, 2, 2, 2, 2, 3)\)
Main results

Set $D_j = jE_0 - \sum_{i=1}^{n} (t + j - 1)E_i$. Imposing that $D_j \cdot E \leq -2$, we obtain:

**Theorem (Harbourne - Schenck- S.)**

Let $I = \langle l_1^t, \ldots, l_n^t \rangle \subseteq \mathbb{K}[x_1, x_2, x_3, x_4] = S$ with $l_i \in S_1$ generic. If $n \in \{5, 6, 7, 8\}$, then WLP fails, respectively, for $t \geq \{3, 27, 140, 704\}$. 
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**Conjecture (Harbourne - Schenck- S.)**

For $I = \langle l_1^t, \ldots, l_n^t \rangle \subseteq \mathbb{K}[x_1, \ldots, x_r] = S$ with $l_i \in S_1$ generic and $n \geq r + 1 \geq 5$, WLP fails for all $t \gg 0$. 