Inverse systems, fat points and the Weak Lefschetz Property

Alexandra Seceleanu

University of Illinois at Urbana-Champaign

AMS National Meeting in New Orleans
January 2011

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_r]$ be an ideal such that $A = S/I$ is Artinian.

Definition

A graded Artinian algebra A has the **Weak Lefschetz Property** *(***WLP***) if there is an element $\ell \in S_1$ such that for all degrees j, the map $\mu_\ell : \mathbf{A_j} \xrightarrow{\cdot \ell} \mathbf{A_{j+1}}$ is either injective or surjective (equivalently μ_ℓ has maximum rank as a $\mathbb{K}$-vector space homomorphism).*

Let $I \subseteq S = \mathbb{K}[x_1, \ldots, x_r]$ be an ideal such that $A = S/I$ is Artinian.

Definition

A graded Artinian algebra A has the **Weak Lefschetz Property** *(***WLP***) if there is an element $\ell \in S_1$ such that for all degrees j, the map $\mu_\ell : \mathbf{A_j} \xrightarrow{\cdot \ell} \mathbf{A_{j+1}}$ is either injective or surjective (equivalently μ_ℓ has maximum rank as a $\mathbb{K}$-vector space homomorphism).*

- The set of linear forms ℓ with this property is Zariski open in S_1.
- We assume henceforth that $\ell \in S_1$ is generic.
- We assume also that $char(\mathbb{K}) = 0$.

WLP is

- known to hold for monomial Artinian complete intersections (Stanley, 1980).
- expected to hold for Artinian ideals generated by *generic* forms.
- known to hold for Artinian ideals generated by *generic* forms in $\mathbb{K}[x_1, x_2, x_3]$ (Anick, 1986).
- known to hold for *any* Artinian ideal generated by powers of linear forms in $\mathbb{K}[x_1, x_2, x_3]$ (Schenck - S., 2009).
- known to hold for Artinian ideals generated by *generic* forms in $\mathbb{K}[x_1, x_2, x_3, x_4]$ (Migliore - Miró-Roig, 2003).

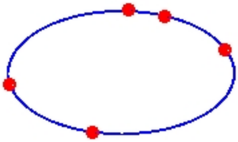
Background

WLP is

- known to hold for monomial Artinian complete intersections (Stanley, 1980).
- expected to hold for Artinian ideals generated by *generic* forms.
- known to hold for Artinian ideals generated by *generic* forms in $\mathbb{K}[x_1, x_2, x_3]$ (Anick, 1986).
- known to hold for *any* Artinian ideal generated by powers of linear forms in $\mathbb{K}[x_1, x_2, x_3]$ (Schenck - S., 2009).
- known to hold for Artinian ideals generated by *generic* forms in $\mathbb{K}[x_1, x_2, x_3, x_4]$ (Migliore - Miró-Roig, 2003).

How about powers of linear forms in $\mathbb{K}[x_1, x_2, x_3, x_4]$?

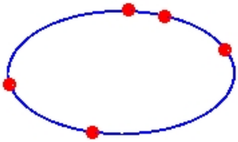
Motivating Example: 5 points on a conic



Look ahead:

- $I = \left(\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3\right) \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]$.

Motivating Example: 5 points on a conic



Look ahead:

- $I = \left(\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3\right) \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]$.
- The space of quartics in $\mathbb{P}^2$ passing through five double points is nonempty $\Longrightarrow$ WLP fails for geometric reasons.

First tool - (Macaulay) inverse systems

Let $\{p_1, \ldots, p_n\} \subseteq \mathbb{P}^{r-1}$ be a set of distinct points defined by ideals $I(p_i) = \wp_i \subseteq R = \mathbb{K}[y_1, \ldots, y_r]$. A **fat point ideal** is an ideal of the form

$$F = \bigcap_{i=1}^{n} \wp_i^{m_i} \subset R.$$

Let $\{p_1, \ldots, p_n\} \subseteq \mathbb{P}^{r-1}$ be a set of distinct points defined by ideals $I(p_i) = \wp_i \subseteq R = \mathbb{K}[y_1, \ldots, y_r]$. A **fat point ideal** is an ideal of the form

$$F = \bigcap_{i=1}^{n} \wp_i^{m_i} \subset R.$$

Recall $S = \mathbb{K}[x_1, \ldots, x_r]$ and define an action of R on S by partial differentiation: $y_j \cdot x_i = \partial x_i / \partial x_j$.

Definition

The set of elements annihilated by the action of F is denoted F^{-1} and called the **(Macaulay) inverse system** *associated to the ideal F.*

Linear forms come into play

Emsalem and Iarrobino proved that there is a close connection between ideals generated by powers of linear forms and ideals of fat points.

Theorem (Emsalem and Iarrobino)

Let F be an ideal of fatpoints:

$$F = \wp_1^{m_1} \cap \cdots \cap \wp_n^{m_n} \subset R.$$

Linear forms come into play

Emsalem and Iarrobino proved that there is a close connection between ideals generated by powers of linear forms and ideals of fat points.

Theorem (Emsalem and Iarrobino)

Let F be an ideal of fatpoints:

$$F = \wp_1^{m_1} \cap \cdots \cap \wp_n^{m_n} \subset R.$$

Then

$$(F^{-1})_j = \begin{cases} S_j & \text{for } j < \max\{m_i\} \\ L_{p_1}^{j-m_1+1} S_{m_1-1} + \cdots + L_{p_n}^{j-m_n+1} S_{m_n-1} & \text{for } j \geq \max\{m_i\} \end{cases}$$

and

$$\dim_{\mathbb{K}}(F^{-1})_j = \dim_{\mathbb{K}}(R/F)_j.$$

Second tool -the syzygy bundle

Harima-Migliore-Nagel-Watanabe have introduced the syzygy bundle as a crucial tool in studying the WLP.

Definition

If $I = \langle f_1, \ldots, f_n \rangle$ *is* $\langle x_1, \ldots, x_r \rangle-$*primary, and* $deg(f_i) = d_i$, *then the* **syzygy bundle** $\mathcal{S}(I) = \widetilde{Syz(I)}$ *is a rank* $n-1$ *bundle defined via*

$$0 \longrightarrow Syz(I) \longrightarrow \bigoplus_{i=1}^{n} S(-d_i) \xrightarrow{[f_1,\ldots,f_n]} S \longrightarrow S/I \longrightarrow 0.$$

Most importantly, $H^1(\mathcal{S}(I)(j)) = A_j$.

WLP and the syzygy bundle

The long exact sequence in cohomology given by the restriction of the syzygy bundle to a hyperplane L defined by the linear form l yields:

$$0 \longrightarrow H^0(\mathcal{S}(I)(j)) \longrightarrow H^0(\mathcal{S}(I)(j+1)) \longrightarrow H^0(\mathcal{S}(I)|_L(j+1)) \longrightarrow$$

$$\longrightarrow A_j \xrightarrow{\cdot \ell} A_{j+1} \longrightarrow H^1(\mathcal{S}(I)|_L(j+1)) \longrightarrow$$

$$\longrightarrow H^2(\mathcal{S}(I)(j)) \longrightarrow \cdots.$$

Divisors on blowups

Consider the fat points ideal $F = \wp_1^{m_1} \cap \cdots \cap \wp_n^{m_n} \subset R$.

On the blowup $\mathbf{X}$ of $\mathbb{P}^{r-1}$ at the points $p_1, \ldots, p_n$, let

- E_i be the class of the exceptional divisor over the point p_i
- E_0 be the pullback of a hyperplane on $\mathbb{P}^{r-1}$

The divisor

$$D_j = jE_0 - \sum_{i=1}^{n} (j - m_i + 1)E_i.$$

describes the global sections of the syzygy bundle

$$h^0(\mathcal{S}(I)(j)) = h^1(D_j)$$

Special divisors in $\mathbb{P}^2$

Definition

A linear system of degree d through a set of fat points $\wp_1, \ldots, \wp_n$ with multiplicities $m_1, \cdots, m_n$ in $\mathbb{P}^2$ is **special** *if its dimension excedes the expected dimension $\binom{d+2}{2} - \sum_{i=1}^{n} \binom{m_i+1}{2} - 1$.*

- E.g. The linear system of quartics through 5 double points has negative expected dimension, but its actual dimension is 1.
- By Riemann-Roch, the space of global sections (or H^0 cohomology) is larger than expected iff the H^1 cohomology $\neq 0$.

Definition

We say $D = dE_0 - \sum_{i=1}^{n} m_i E_i$ is **special** *if $h^0(D)$ and $h^1(D)$ are positive.*

Motivating Example revisited

Let $I = \left(\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3\right) \subset S = \mathbb{K}[x_1, x_2, x_3, x_4]$ and let $A = S/I$.

The Hilbert function of A is:

j	0	1	2	3	4	5	6	...
$\dim_{\mathbb{K}} A_j$	1	4	10	15	15	6	0	...

Motivating Example revisited

Let $I = \left(\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3\right) \subset S = \mathbb{K}[x_1, x_2, x_3, x_4]$ and let $A = S/I$.

The Hilbert function of A is:

j	0	1	2	3	4	5	6	...
$\dim_{\mathbb{K}} A_j$	1	4	10	15	15	6	0	...

$$0 \longrightarrow H^0(\mathcal{S}(I)(3) \longrightarrow H^0(\mathcal{S}(I)(4)) \longrightarrow H^0(\mathcal{S}(I)|_L(4)) \longrightarrow$$

$$\longrightarrow A_3 \xrightarrow{\cdot \ell} A_4 \longrightarrow H^1(\mathcal{S}(I)|_L(4)) \longrightarrow$$

$$\longrightarrow H^2(\mathcal{S}(I)(3)) \longrightarrow \cdots .$$

Let $I = \left(\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3\right) \subset S = \mathbb{K}[x_1, x_2, x_3, x_4]$ and let $A = S/I$.

The Hilbert function of A is:

j	0	1	2	3	4	5	6	...
$\dim_{\mathbb{K}} A_j$	1	4	10	15	15	6	0	...

$$0 \longrightarrow H^0(\mathcal{S}(I)(3) \longrightarrow \overset{0}{H^0(\mathcal{S}(I)(4))} \longrightarrow \overset{1}{H^0(\mathcal{S}(I)|_L(4))}$$

$$\longrightarrow \overset{15}{A_3} \xrightarrow{\cdot \ell} \overset{15}{A_4} \longrightarrow \overset{1}{H^1(\mathcal{S}(I)|_L(4))}$$

$$\longrightarrow \overset{0}{H^2(\mathcal{S}(I)(3))} \longrightarrow \cdots.$$

Special divisors and (-1)-curves in $\mathbb{P}^2$

Conjecture (Segre-Harbourne-Gimigliano-Hirschowitz SHGH)

If $D = dE_0 - \sum_{i=1}^{n} m_i E_i$ is a special divisor on a blowup of $\mathbb{P}^2$, then there exists a (-1)-curve E with $E \cdot D \leq -2$. ((-1)-curve means $E \cdot E = -1$)

- This conjecture is known to be true for $n \leq 8$ points.

Special divisors and (-1)-curves in $\mathbb{P}^2$

Conjecture (Segre-Harbourne-Gimigliano-Hirschowitz SHGH)

If $D = dE_0 - \sum_{i=1}^{n} m_i E_i$ is a special divisor on a blowup of $\mathbb{P}^2$, then there exists a (-1)-curve E with $E \cdot D \leq -2$. ((-1)-curve means $E \cdot E = -1$)

- This conjecture is known to be true for $n \leq 8$ points.

Theorem (S.)

If $E = dE_0 - \sum_{i=1}^{8} m_i E_i$ is the divisor of a (-1)-curve on a blowup of $\mathbb{P}^2$ at $n \leq 8$ points, then the coefficients are given by

- $d = 0$, $m_i = (-1, 0, 0, 0, 0, 0, 0, 0)$
- $d = 1$, $m_i = (0, 0, 0, 0, 0, 0, 1, 1)$
- $d = 2$, $m_i = (0, 0, 0, 1, 1, 1, 1, 1)$
- $d = 3$, $m_i = (0, 1, 1, 1, 1, 1, 1, 2)$
- $d = 4$, $m_i = (1, 1, 1, 1, 1, 2, 2, 2)$
- $d = 5$, $m_i = (1, 1, 2, 2, 2, 2, 2, 2)$
- $d = 6$, $m_i = (2, 2, 2, 2, 2, 2, 2, 3)$

Set $D_j = jE_0 - \sum_{i=1}^{n}(t+j-1)E_i$. Imposing that $D_j \cdot E \leq -2$, we obtain:

Theorem (Harbourne - Schenck- S.)

Let $I = \langle l_1^t, \ldots, l_n^t \rangle \subseteq \mathbb{K}[x_1, x_2, x_3, x_4] = S$ *with* $l_i \in S_1$ *generic. If* $n \in \{5, 6, 7, 8\}$*, then WLP fails, respectively, for* $t \geq \{3, 27, 140, 704\}$*.*

Main results

Set $D_j = jE_0 - \sum_{i=1}^{n}(t+j-1)E_i$. Imposing that $D_j \cdot E \leq -2$, we obtain:

Theorem (Harbourne - Schenck- S.)

Let $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq \mathbb{K}[x_1, x_2, x_3, x_4] = S$ *with* $l_i \in S_1$ *generic. If* $n \in \{5, 6, 7, 8\}$, *then WLP fails, respectively, for* $t \geq \{3, 27, 140, 704\}$.

Conjecture (Harbourne - Schenck- S.)

For $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq \mathbb{K}[x_1, \dots, x_r] = S$ *with* $l_i \in S_1$ *generic and* $n \geq r+1 \geq 5$, *WLP fails for all* $t \gg 0$.