Inverse systems, fat points and the Weak Lefschetz Property

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Goal

Give an overview of open problems and known results on the Weak and Strong Lefschetz properties, with an emphasis on the vast number of different approaches and tools that have been used.

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Then describe recent work with Hal Schenck (UIUC) and Brian Harbourne (UNL) on ideals generated by powers of linear forms.

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- $R = k[x_1, ..., x_r]$ graded polynomial ring.
- Let $A = R/I = \bigoplus_{i=0}^{n} A_i$ be a graded Artinian algebra. Note that A is finite dimensional over k.

The Weak Lefschetz Property

Let ℓ be a general linear form.

For each i, ℓ induces a homomorphism $A_i \xrightarrow{\cdot\ell} A_{i+1}$.

We expect maximum rank:

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The set of linear forms ℓ for which the map $\cdot\ell$ has maximum rank is an open set in R_1, hence the use of the term "general linear form". But is this open set non-empty?

Definition

Let ℓ be a general linear form. We say that A has the **Weak Lefschetz Property (WLP)** *if*

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- Note that ℓ^d is not a general form of degree d.

Example

The artinian algebra $A = k[x,y,z]/(x^3, y^3, z^3, (x+y+z)^3)$

- has the WLP;
- does NOT have the SLP because $(\cdot \ell^3)$ fails to have maximal rank.

Theorem (Harima, Migliore, Nagel and Watanabe (2003))

If $char(k) = 0$ and I is any homogeneous ideal in $k[x, y]$ then R/I has the SLP.

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Remark:
It is not hard to show that the above theorem is true for WLP in any characteristic. ([Migliore-Zanello [2007], Li-Zanello [2011], Cook-Nagel [2011p].)

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Theorem (Stanley (1980), J. Watanabe (1987), Reid-Roberts-Roitman (1991))

Let $R = k[x_1, ..., x_r]$, where k has characteristic zero. Let I be an Artinian monomial complete intersection, i.e.

$$I = (x_1^{a_1}, \ldots, x_r^{a_r})$$

Then R/I has the SLP.

In particular, R/I has the WLP.

Open problems (1)

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   Whether artinian Gorenstein algebras in three variables have the WLP or SLP is still wide open, and one of my favorite open problems.

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Main results

Theorem (Schenck- S.)

Let $I = \langle l_1^{a_1}, \ldots, l_n^{a_n} \rangle \subseteq k[x_1, x_2, x_3]$ *for* any $l_i \in S_1$, *then WLP holds for* R/I.

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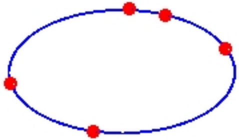
Theorem (Harbourne - Schenck- S.)

Let $I = \langle l_1^t, \ldots, l_n^t \rangle \subseteq k[x_1, x_2, x_3, x_4] = S$ *with* $l_i \in S_1$ generic*. If* $n \in \{5, 6, 7, 8\}$*, then WLP fails for* R/I*, respectively, for* $t \geq \{3, 27, 140, 704\}$*.*

Motivating example



Look ahead:

- $I = \left(\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3\right) \subseteq \mathbb{K}[x_1, x_2, x_3, x_4].$

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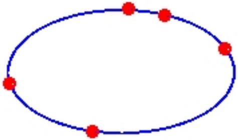

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- The space of quartics in $\mathbb{P}^2$ passing through five double points is nonempty $\implies$ WLP fails for geometric reasons.

First tool -the syzygy bundle

Harima-Migliore-Nagel-Watanabe have introduced the syzygy bundle as a crucial tool in studying the WLP.

Definition

If $I = \langle f_1, \ldots, f_n \rangle$ is $\langle x_1, \ldots, x_r \rangle$−primary, and $deg(f_i) = d_i$, then the **syzygy bundle** $\mathcal{S}(I) = \widetilde{Syz(I)}$ *is a rank $n-1$ bundle defined via*

$$0 \longrightarrow Syz(I) \longrightarrow \bigoplus_{i=1}^{n} R(-d_i) \xrightarrow{[f_1,\ldots,f_n]} R \longrightarrow R/I \longrightarrow 0.$$

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$$0 \longrightarrow \mathcal{S}(I) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^{r-1}}(-d_i) \xrightarrow{[f_1,\dots,f_n]} \mathcal{O}_{\mathbb{P}^{r-1}} \longrightarrow 0.$$

Most importantly, $H^1(\mathcal{S}(I)(j)) = A_j$.

WLP and the syzygy bundle

The long exact sequence in cohomology given by the restriction of the syzygy bundle to a hyperplane L defined by the linear form l yields:

$$0 \longrightarrow H^0(\mathcal{S}(I)(j)) \longrightarrow H^0(\mathcal{S}(I)(j+1)) \longrightarrow H^0(\mathcal{S}(I)|_L(j+1)) \longrightarrow A_j \xrightarrow{\cdot \ell} A_{j+1} \longrightarrow H^1(\mathcal{S}(I)|_L(j+1)) \longrightarrow H^2(\mathcal{S}(I)(j)) \longrightarrow \cdots.$$

Second tool - (Macaulay) inverse systems

Let $R = K[x_1, \dots x_n]$ and $S = K[y_1, \dots, y_n]$ a new ring.
Define an action of S on R by partial differentiation

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For any ideal $I \subset S$ we define the following S-submodule of R, which we call the inverse system of I:

$$I^{-1} := \{g \in R \mid \langle f, g \rangle = 0 \ \ \forall f \in I \ \} = \{g \in R \mid I \circ g = 0\}.$$

fat points

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Theorem (Emsalem and Iarrobino)

Inverse systems describe a duality between the fat points and the corresponding ideal defined by powers of linear forms:

$$(F^{-1})_j = I \text{ and } HF(S/F, j) = HF(I, j).$$

Divisors on blowups

Consider the fat points ideal $F = \wp_1^{m_1} \cap \cdots \cap \wp_n^{m_n} \subset S$.

On the blowup $\mathbf{X}$ of $\mathbb{P}^{r-1}$ at the points $p_1, \ldots, p_n$, let

- E_i be the class of the exceptional divisor over the point p_i
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The divisor

$$D_j = jE_0 - \sum_{i=1}^{n} (j - m_i + 1)E_i.$$

describes the global sections of the syzygy bundle

$$h^0(\mathcal{S}(I)(j)) = h^1(D_j)$$

Special divisors in $\mathbb{P}^2$

Definition

A linear system of degree d through a set of fat points $\wp_1, \ldots, \wp_n$ with multiplicities $m_1, \cdots, m_n$ in $\mathbb{P}^2$ is **special** *if its dimension excedes the expected dimension $\binom{d+2}{2} - \sum_{i=1}^{n} \binom{m_i+1}{2} - 1$.*

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- DeVolder-Laface: give numerical conditions for a divisor on a blowup of $\mathbb{P}^3$ to be non-special.

Motivating Example revisited

Let $I = (\ell_1^3, \ell_2^3, \ell_3^3, \ell_4^3, \ell_5^3) \subset S = k[x_1, x_2, x_3, x_4]$ and let $A = S/I$.

The Hilbert function of A is:

j	0	1	2	3	4	5	6	...
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Conversely, some Hilbert functions force the WLP these were classified in [MZ 2007].

Relations with Fröberg's Conjecture

Thinking of $I = (\ell_1^3, \dots, \ell_5^3)$ from our motivating example, notice that $R/(I, \ell) = S/J$, where J is the ideal of cubes of five general linear forms in $k[x, y, z]$. Thus $\dim[S/J]_4 = 1$. Let K be the ideal of five general cubics in S. Fröberg predicts (and Anick proves) that $\dim[S/K]_4 = 0$.

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In fact, Migliore-Miro-Roig-Nagel prove that whenever an ideal of n powers of general linear forms fails the WLP (for specified exponents), then for some subset of these powers of general linear forms, the same number and powers of general linear forms in one fewer variable fails to have Fröbergs predicted Hilbert function.

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