1 Motivation

We begin by recalling the definition of WLP:

**Definition 1.** Let \( I \subseteq S = K[x_1, \ldots, x_n] \) be an ideal such that \( A = S/I \) is Artinian. Then \( A \) has the Weak Lefschetz property (WLP) if there is an \( \ell \in S_1 \) such that for all \( m \), the map \( \mu_\ell : A_m \to A_{m+1} \) is either injective or surjective.

To study the WLP one is motivated by the interest in determining the Hilbert functions of ideals generated by generic forms.

In the following we fix \( I(\alpha_1, \ldots, \alpha_r) = (L_{\alpha_1}^{\alpha_1}, \ldots, L_{\alpha_r}^{\alpha_r}) \).

2 Inverse systems

In [EI95], Emsalem and Iarrobino proved that there is a close connection between ideals generated by powers of linear forms, and ideals of fatpoints.

Let \( R = K[y_1, \ldots, y_n], p_i = [p_{i1} : \cdots : p_{im}] \in \mathbb{P}^{n-1}, I(p_i) = \wp_i \). A fat point ideal is an ideal of the form

\[
F = \bigcap_{i=1}^{r} \wp_{i}^{\alpha_i+1} \subset R.
\]

Define an action of \( R \) on \( S \) by partial differentiation: \( y_j = \partial/\partial x_j \). Since \( F \) is a submodule of \( R \), it acts on \( S \). The set of elements annihilated by the action of \( F \) is denoted \( F^{-1} \). Let \( L_{p_i} = \sum_{j=1}^{n} p_{ij} x_j \in S \). Emsalem and Iarrobino show

**Theorem 2** (Emsalem and Iarrobino, [EI95]). *Let \( F \) be an ideal of fatpoints:*

\[
F = \wp_{1}^{\alpha_1+1} \cap \cdots \cap \wp_{r}^{\alpha_r+1} \subset R.
\]

*Then*

\[
(F^{-1})_j = \begin{cases} 
S_j & \text{for } j \leq \max \{\alpha_i\} \\
L_{p_1}^{j-\alpha_1} S_{\alpha_1} + \cdots + L_{p_r}^{j-\alpha_r} S_{\alpha_r} & \text{for } j \geq \max \{\alpha_i + 1\}
\end{cases}
\]

*and*

\[
HF(F^{-1}, j) = HF\left( R \frac{R}{I(j - \alpha_1, \ldots, j - \alpha_r)}, j \right).
\]
We are trying to use the correspondence in the "reverse direction". Note that to obtain the Hilbert function of a fixed ideal of linear forms, it is necessary to consider an infinite family of ideals of fat points.

3 Powers of linear forms and blowing up projective space

There is a well-known correspondence between the graded pieces of an ideal of fat points $F \subseteq K[x_1, \ldots, x_r]$ and the global sections of a line bundle on the variety $X$ which is the blow up of $\mathbb{P}^{r-1}$ at the points. We briefly review this. Let $\mathcal{I}_Z(j)$ be the ideal sheaf of the fat points subscheme $Z$ defined by $F$. Of course, $\dim_K F_j = h^0(\mathbb{P}^{r-1}, \mathcal{I}_Z(j))$.

Let $E_i$ be the class of the exceptional divisor over the point $p_i$, and $E_0$ the pullback of a hyperplane on $\mathbb{P}^{r-1}$. Define

$$D_j = jE_0 - \sum_{i=1}^n m_i E_i.$$  

Moreover, $h^i(X, D) = h^i(\mathbb{P}^{r-1}, \mathcal{I}_Z(j))$ for all $i \geq 0$. Taking cohomology of the exact sequence

$$0 \rightarrow \mathcal{I}_Z(j) \rightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(j) \rightarrow \mathcal{O}_Z \rightarrow 0$$

and using the fact that $\mathcal{O}_Z(j) \cong \mathcal{O}_Z$ and thus $h^0(Z, \mathcal{O}_Z(j)) = h^0(Z, \mathcal{O}_Z) = \sum_i \binom{n-2+m_i}{n-1}$, shows that

$$h^0(X, D) = h^0(\mathcal{I}_Z(j)) = \binom{n-1+j}{n-1} - \sum_i \binom{n-2+m_i}{n-1} + h^1(\mathcal{I}_Z(j)).$$  \hfill (3.1)

In the context of Theorem 2, taking $m_i = j - t + 1$ for all $i$ and defining $D_j$ to be $D_j = jE_0 - (j - t + 1)(E_1 + \cdots + E_n)$, we thus have:

$$\dim_K I_j = \begin{cases} n\binom{r+j-t-1}{r-1} - h^1(\mathcal{I}_Z(j)) = n\binom{r+j-t-1}{r-1} - h^1(D_j) & \text{for } j \geq t \\ 0 & \text{for } 0 \leq j < t \end{cases}$$  \hfill (3.2)

Alternatively, this can be stated for the quotient $S/I = A$ as:

$$\dim_K A_j = \begin{cases} h^0(D_j) & \text{for } j \geq t \\ \binom{r-1+j}{r-1} & \text{for } 0 \leq j < t \end{cases}$$  \hfill (3.3)
**Definition 3.** We will say that $I$ has expected dimension in degree $j$ if either $I_j = 0$ or $h^1(D_j) = 0$. We say $D_j$ is irregular if $h^1(D_j) > 0$ and regular otherwise. We say $D_j$ is special if $h^0(D_j)$ and $h^1(D_j)$ are both positive.

A landmark result on the dimension of linear systems is:

**Theorem 4 (Alexander–Hirschowitz [AH92]).** Fix $m, n - 1 \geq 2$, and consider the linear system of hypersurfaces of degree $m$ in $\mathbb{P}^{n-1}$ passing through $n$ general points with multiplicity two. Then

1. For $m = 2$, the system is special iff $2 \leq r \leq n - 1$.

2. For $m$ greater than two, the only special systems are $(n - 1, m, r) \in \{(2, 4, 5), (3, 4, 9), (4, 4, 14), (4, 3, 7)\}$.

In each of these four cases, the linear system is expected to be empty but in fact has projective dimension 0.

**Example 5.** Let $A$ be the quotient of $K[x_1, x_2, x_3]$ by the cubes of five general linear forms. The corresponding five points in $\mathbb{P}^2$ are general, and the first interesting computation involves $D_4 = 4E_0 - \sum_{i=1}^{5} 2E_i$, for which we have

$$\dim K A_4 = h^0(D_4) = \binom{6}{2} - 15 + h^1(D_4).$$

Since $H^0(D_4)$ contains the double of a conic through the five points, $D_4$ is special, and in fact we have $h^0(D_4) = 1 = h^1(D_4)$.

A famous open conjecture on the Hilbert function of fat points in $\mathbb{P}^2$ is expressed in terms of $(-1)$-curves (i.e., smooth rational curves $E$ with $E^2 = -1$):

**Conjecture 6 (Segre-Harbourne-Gimigliano-Hirschowitz [?]).** Suppose that $\{p_1, \ldots, p_n\} \subseteq \mathbb{P}^2$ is a collection of points in general position, $X$ is the blowup of $\mathbb{P}^2$ at the points, and $E_i$ the exceptional divisor over $p_i$. If $F_j = jE_0 - \sum_{i=1}^{n} a_i E_i$ is special, then there exists a $(-1)$-curve $E$ with $E \cdot F_j \leq -2$.

**Example 7.** Let $C = 2(2E_0 - \sum_{i=1}^{5} E_i) + (E_0 - E_1 - E_2)$. Then $h^0(C) = 1$ and $h^1(C) = 1$, so $C$ is special, but $E = 2E_0 - \sum_{i=1}^{5} E_i$ is rational by adjunction with $E^2 = -1$ and $E \cdot C = -2$. 


4 Main results

The case of ideals generated by powers of linear forms in $K[x_1,\ldots, x_3]$ has been answered in the affirmative in previous work [?].

Recall that one of the strategies (described in previous talk) of analyzing the WLP is through the syzygy bundle of the associated ideal sheaf restricted to a line bundle. In this section, we focus on powers of linear forms in $S=K[x_1,\ldots, x_4]$ for which the associated (restricted) fat point subscheme is a subset of $\mathbb{P}^2$ and we furthermore restrict to fat point schemes whose Hilbert function is known. This means looking at fat point schemes supported at 8 or less points.

Recall that via the syzygy bundle techniques one obtains a long exact sequence in cohomology

$$0 \longrightarrow H^0(S(I)(m)) \longrightarrow H^0(S(I)(m+1)) \xrightarrow{\phi_m} H^0(S(I)|_L(m+1)) \longrightarrow$$

$$\xrightarrow{\psi_m} H^1(S(I)(m)) \longrightarrow H^1(S(I)(m+1)) \xrightarrow{\mu} H^1(S(I)|_L(m+1))$$

$$\xrightarrow{\psi_m} H^2(S(I)(m)) \longrightarrow H^2(S(I)(m+1)) \longrightarrow \cdots .$$

(4.1)

We observed that surjectivity of the map $\phi_m$ implies injectivity of the WLP map and injectivity of $\psi_m$ implies surjectivity of the WLP map. The present work translates these observations into the language of divisors on the blowup of $\mathbb{P}^2$.

**Theorem 8.** If the divisor $D_m$ (as defined in the previous section) on the blowup of $\mathbb{P}^n$ is non-special and the divisor $D_{m+1}$ on $\mathbb{P}^{n-1}$ is special then WLP fails for the corresponding ideal generated by powers of linear forms.

**Example 9.** Here we apply the theorem to obtain the Hilbert function for $A=K[x_1,x_2,x_3,x_4]/\langle x_1^3, x_2^3, x_3^3, x_4^3, (x_1+x_2+x_3+x_4)^3 \rangle$.

<table>
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<th>$j$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\dim_K A_j$</td>
<td>$1$</td>
<td>$4$</td>
<td>$10$</td>
<td>$15$</td>
<td>$15$</td>
<td>$6$</td>
<td>$0$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$HF(\cap_{i=1}^5 \varphi_i^{j-2}, j)$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$15$</td>
<td>$15$</td>
<td>$6$</td>
<td>$0$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

We consider the restriction of this example to $\mathbb{P}^2$ in Example 5.

**Example 10.** But as in the proof of Proposition $??(c)$, the kernel of $A_3 \rightarrow A_4$ has dimension $h^1(D'_4)$, hence the cokernel has dimension $h^1(D'_4)$, so $\mu$ fails to have full rank, since $h^1(D'_4) = 1$ by Theorem 4.
We employ this theorem together with the Alexander-Hirschowitz (for 5 points) and Segre-Harbourne-Gimigliano-Hirschowitz Theorem to fully describe the failure of the WLP in the case of 5, 6, 7, 8 fat points.

**Theorem 11.** Let \( I = \langle l_1, \ldots, l_n \rangle \subseteq K[x_1, x_2, x_3, x_4] \) with \( l_i \in S_1 \) generic. If \( n \in \{5, 6, 7, 8\} \), then WLP fails, respectively, for \( t \geq \{3, 27, 140, 704\} \).

5 Connections with Gelfand-Tsetlin patterns

**Definition 12.** A two-row Gelfand-Tsetlin pattern is a non-negative integer \( 2 \times n \)-matrix \((\lambda_{ij})\) that satisfies \( \lambda_{2n} = 0 \), \( \lambda_{1,j+1} \geq \lambda_{2,j} \) and \( \lambda_{i,j} \geq \lambda_{i,j+1} \) for \( i = 1, 2 \) and \( j = 1, \ldots, n - 1 \).

In Proposition 3.6 of [SX10], Sturmfels-Xu show that for generic forms \( l_i \), the Hilbert function of \( K[x_1, \ldots, x_r]/\langle l_{u_1}, \ldots, l_{u_{r+1}} \rangle \) in degree \( i \) is the number of two-rowed Gelfand-Tsetlin patterns with \( \lambda_{21} = i \) and \( \lambda_{1j} + \lambda_{2j} = u_j + \cdots + u_{r+1} \) for \( j = 1, \ldots, r + 1 \).

References


