Using syzygies to test containments between ordinary and symbolic powers

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Symbolic powers of ideals

The n-th **symbolic power** $I^{(n)}$ of an ideal $I \subset R$ is

$$I^{(n)} = \bigcap_{P \in \mathrm{Min(I)}} I^n R_P \cap R.$$

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In a geometric sense, symbolic powers have an particularly nice meaning:

- Zariski-Nagata: if $\mathbf{X} = V(I)$ is an algebraic variety, then $I^{(n)}$ is the set of forms that vanish to order at least n at every point of $\mathbf{X}$
- in characteristic 0, $I^{(n)} =$ the forms that vanish together with their first $n-1$ partial derivatives at every point of $\mathbf{X}$.

Comparing symbolic and ordinary powers

Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)

For any homogeneous ideal $I \subseteq K[\mathbb{P}^N] = K[x_0, \ldots, x_N]$*, the following containment holds*

$$\mathbf{I}^{(\mathbf{Nr})} \subseteq \mathbf{I}^{\mathbf{r}}, \forall r \geq 1$$

proven by

- Ein-Lazarsfeld-Smith (2001), for I unmixed, using multiplier ideals
- Hochster-Huneke (2002) using reduction to characteristic p and tight closure

Improving the containment

From now on let I be an ideal defining points in $\mathbb{P}^N$.

- ELS–HH: $I^{(Nr)} \subseteq I^r, \forall r \geq 1 \Rightarrow$ Waldschmidt-Skoda: $\frac{\alpha(I^{(m)})}{m} \geq \frac{\alpha(I)}{N}$

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Can make containment tighter in 2 ways. Each leads to a conjecture:

- Conjecture 1: (Harbourne-Huneke)
  $I^{(Nr-N+1)} \subseteq I^r$ for all $r \geq 1$ and all ideals I defining points in $\mathbb{P}^N$.
- Conjecture 2: (Harbourne-Huneke) $\Rightarrow$ Chudnowsky's Conjecture
  $I^{(Nr)} \subseteq \mathfrak{m}^{(N-1)r} I^r$ for all $r \geq 1$ and all ideals I defining points in $\mathbb{P}^N$.

The case $N = 2$, $r = 2$

For $N = 2$, $r = 2$, the ELS–HH theorem states that $I^{(4)} \subseteq I^2$.

Question (Huneke)

Does

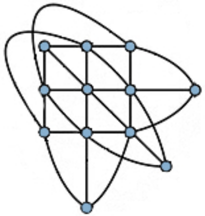
$$I^{(3)} \subseteq I^2$$

always hold in the case of I defining a reduced set of points of $\mathbb{P}^2$?

- Bocci-Harbourne: $I^{(3)} \subseteq I^2$ holds for points in general position in $\mathbb{P}^2$.

Three configurations

Fermat configuration



12 pts & 9 lines
12 triple pts

realizable over
fields containing ξ
$\xi^3 = 1$

Klein configuration

49 pts & 21 lines
21 quadruple
28 triple

realizable over

$K[a]/(a^2 + a + 2)$
e.g. $\mathbb{R}[\sqrt{-7}]$
or $\mathbb{Z}/7$

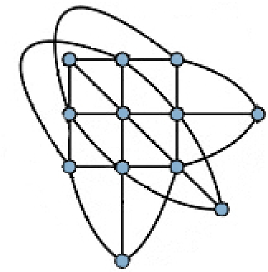
Wiman configuration

201 pts & 45 lines
36 quintuple
45 quadruple
120 triple

realizable over

$K[a]/(a^4 - a^2 + 4)$
e.g. $\mathbb{Z}/19$
or $\mathbb{Z}/31$

Geometric intuition



Let F =the product of all the lines in the configuration

- $F \in I^{(3)}$ is easy to see: every point is a triple point
- $F \notin I^2$ is more challenging to prove

This is a common feature of the 3 counterexamples.

A homological criterion

Basic idea: certainly $I^3 \subseteq I^2$

- $\mathbf{I^{(3)} \subseteq I^2} \iff H^0_m(R/I^3) = \frac{I^{(3)}}{I^3} \to H^0_m(R/I^2) = \frac{I^{(2)}}{I^2}$ is the zero map.

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- consider a 3-generated ideal $I = (f, g, h) \subseteq k[x, y, z], \mathrm{ht(I)} = 2$
- look at the resolutions of I^2, I^3 to interpret the map above

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^3 & \xrightarrow{Y} & R^{12} & \longrightarrow & R^{10} & \longrightarrow & I^3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R & \longrightarrow & R^6 & \longrightarrow & R^6 & \longrightarrow & I^2 & \longrightarrow & 0 \end{array}$$

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- consider a 3-generated ideal $I = (f, g, h) \subseteq k[x, y, z], \mathrm{ht(I)} = 2$
- look at the resolutions of I^2, I^3 to interpret the map above
- dualize and look at the map $Ext^3(R/I^2, R) \to Ext^3(R/I^3, R)$

$$\begin{array}{ccccccccccc}
0 & \longleftarrow & R^3 & \xleftarrow{Y^T} & R^{12} & \longleftarrow & R^{10} & \longleftarrow & I^3 & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle [f\ g\ h]} & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & R & \longleftarrow & R^6 & \longleftarrow & R^6 & \longleftarrow & I^2 & \longleftarrow & 0
\end{array}$$

Main result

Theorem (S., 2014)

If $I = (f, g, h)$ has minimal generators of the same degree d and defines a reduced set of points in $\mathbf{P}^2$, then:

- *the minimal free resolution of I^3 has the form*

$$0 \longrightarrow R^3 \xrightarrow{\mathbf{Y}} R^{12} \longrightarrow R^{10} \longrightarrow I^3 \longrightarrow 0,$$

(*and* **Y** *can be expressed explicitly in terms of the syzygies of* I)

- *criterion for containment*

$$\mathbf{I}^{(3)} \subseteq \mathbf{I}^{2} \quad \Longleftrightarrow \quad \begin{bmatrix} f \\ g \\ h \end{bmatrix} \in \mathrm{Image}(\mathbf{Y}^{\mathbf{T}}).$$

Application to known counterexamples

Theorem (S., 2014)

If I is any one of the ideals defining the Fermat, Klein or Wiman configurations, then

$$\mathbf{I}^{(3)} \not\subseteq \mathbf{I}^2, \textit{ since } \begin{bmatrix} f \\ g \\ h \end{bmatrix} \notin \mathrm{Image}(\mathbf{Y}^{\mathbf{T}}).$$

Ideas of proof:

- play off the symmetry in the generators f, g, h of I against the symmetry exhibited by the matrix $\mathbf{Y}$ in the resolution of I^3
- e.g for Fermat:

$$\begin{bmatrix} f \\ g \\ h \end{bmatrix} = \begin{bmatrix} zx^3 - zy^3 \\ yx^3 - yz^3 \\ xy^3 - xz^3 \end{bmatrix} \text{ vs } \mathbf{Y} = \begin{pmatrix} xy & 0 & 0 & xz & yz & 0 & -z^2 & 0 & 0 & -y^2 & -x^2 & 0 \\ 0 & xz & 0 & xy & 0 & yz & 0 & -y^2 & 0 & -z^2 & 0 & -x^2 \\ 0 & 0 & yz & 0 & xy & xz & 0 & 0 & -x^2 & 0 & -z^2 & -y^2 \end{pmatrix}$$

Open questions

Revised versions of C. Huneke's question:

- Is it always true for the ideal I of a finite set of points in $\mathbb{P}^2$ that

$$I^{(5)} \subseteq I^3 \text{ ?}$$

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