A hands-on approach to tensor product surfaces

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Surface modeling

Spline surfaces closely following a control grid embedded in $\mathbb{R}^3$ are standard tools of today's CAD systems.

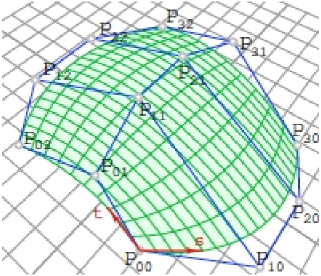


Figure: A surface following a control point grid

Tensor product surface

Definition

A tensor product surface of bidegree (d_1, d_2) is a piecewise polynomial parametric surface surface that is built by connecting several polynomial patches in a smooth manner:

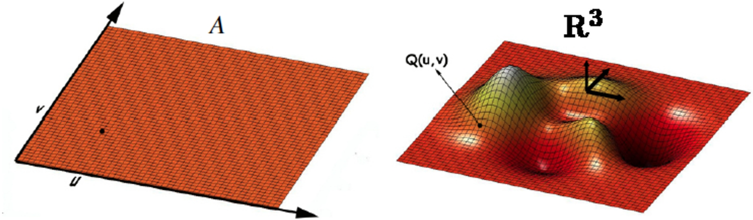
$$f : [0,1] \times [0,1] \longrightarrow \mathbb{A}^3$$

$$(u, s) \mapsto \sum_{i,j=1}^{t} P_{ij} B_i(u) B'_j(s)$$

The control points $P_{ij} \in \mathbb{A}^3$ define the control grid of the spline surface. The polynomials $B_i(u), B'_j(s)$ are called blending functions.

Tensor product surface - homogeneous version

- Instead of thinking about tensor product surfaces as affine maps:



- we homogenize to get a regular map defined by four polynomials with no common zeros on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$((s,t),(u,v)) \mapsto (p_0(s,t,u,v) : p_1(s,t,u,v) : p_2(s,t,u,v) : p_3(s,t,u,v))$$

Main problems in geometric modeling

1. **Implicitize**: We assume the parametrization ϕ known but the implicit equation F of the surface not known!
2. **Find singular locus**: In general one does not want singular points in the interior of the patch, but singular points on the surface decrease the degree of the implicit equation.

Main idea:

- study these problem using the information contained in the resolution of the ideal $I = (p_0 : p_1 : p_2 : p_3)$.

Example

$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3, \ ((s,t),(u,v)) \mapsto (s^2u, s^2v, t^2u, t^2v + stv)$

$I = (s^2u, s^2v, t^2u, t^2v + stv)$

- the **bigraded resolution**

$$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{array}{c} R(-2,-2) \\ \oplus \\ R(-3,-2)^2 \\ \oplus \\ R(-4,-1)^2 \end{array} \leftarrow R(-4,-2)^2 \leftarrow 0$$

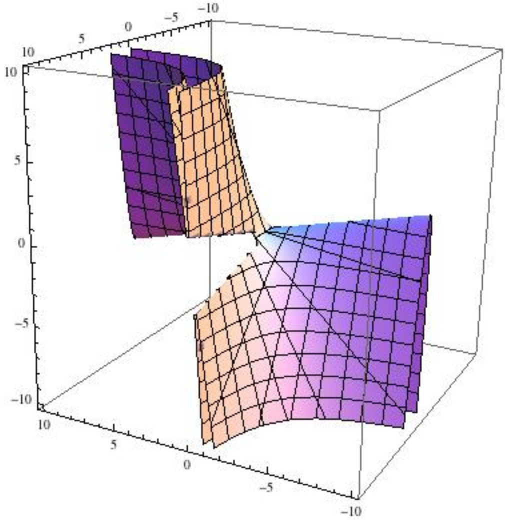
- the **implicit equation**:

$$X = \mathbf{V}(x_0x_1^2x_2 - x_1^2x_2^2 + 2x_0x_1x_2x_3 - x_0^2x_3^2).$$

- the reduced codimension one **singular locus** of X is:

$$\mathbf{V}(x_0, x_2) \cup \mathbf{V}(x_1, x_3) \cup \mathbf{V}(x_0, x_1).$$

The surface in the example



The Segre-Veronese variety

From now on I = four-generated bigraded ideal of bidegree (2,1).

Main idea:

- study the resolution of I by inspecting how the plane

$$\mathrm{Span}(p_0, p_1, p_2, p_3) \subset \mathrm{Span}(s^2u, s^2v, stu, stv, t^2u, t^2v) = \mathbb{P}^5$$

meets the image $\Sigma_{2,1}$ of the Segre map

$$\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(2))) \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(1))) \longrightarrow \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,1)))$$

The Segre-Veronese variety

Main idea:

- study the resolution of I by inspecting how the plane $\mathbb{P}(U) = \mathrm{Span}(p_0, p_1, p_2, p_3)$ meets the image $\Sigma_{2,1}$ of the Segre map
$$\mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{\sigma_{2,1}} \mathbb{P}^5.$$

Recall at each point of $\Sigma_{2,1}$ there are 2 types of fibers:

- lines $\mathbb{P}^1$
- planes $\mathbb{P}^2$

Linear syzygies and lines on $\Sigma_{2,1}$

$\mathbb{P}(U) = \mathrm{Span}(p_0, p_1, p_2, p_3)$.

Proposition

The ideal $I = (p_0, p_1, p_2, p_3)$

1. *has a unique linear syzygy of bidegree* $(0, 1)$ *iff* $F \subseteq \mathbb{P}(U) \cap \Sigma_{2,1}$, *where* F *is a* $\mathbb{P}^1$ *fiber of* $\Sigma_{2,1}$.
2. *has a pair of linear syzygies of bidegree* $(0, 1)$ *iff* $\mathbb{P}(U) \cap \Sigma_{2,1} = \Sigma_{1,1}$.
3. *has a unique linear syzygy of bidegree* $(1, 0)$ *iff* $F \subseteq \mathbb{P}(U) \cap Q$, *where* F *is a* $\mathbb{P}^1$ *fiber of* Q.

Degree (0,2) syzygies and quadrics on $\Sigma_{2,1}$

Proposition

There is a minimal first syzygy on I *of bidegree* $(0,2)$ *iff there exists* $\mathbb{P}(W) \simeq \mathbb{P}^2 \subseteq \mathbb{P}(U)$ *such that* $\mathbb{P}(W) \cap \Sigma_{2,1}$ *is a smooth conic.*

Degree (0,2) syzygies and quadrics on $\Sigma_{2,1}$

Proposition

There is a minimal first syzygy on I of bidegree $(0,2)$ iff there exists $\mathbb{P}(W) \simeq \mathbb{P}^2 \subseteq \mathbb{P}(U)$ such that $\mathbb{P}(W) \cap \Sigma_{2,1}$ is a smooth conic.

Proposition

Only minimal first syzygies of bidegrees $(1,0)$ and $(0,2)$ are compatible.

Main result

Theorem

There are exactly 6 resolutions for ideals generated by four bidegree (2,1) forms without basepoints :

The six Betti types

Type	Bigraded Minimal Free Resolution of I
6	$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{matrix} R(-2,-2)^2 \\ \oplus \\ R(-4,-1)^2 \end{matrix} \leftarrow R(-4,-2) \leftarrow 0$
5	$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{matrix} R(-2,-2) \\ \oplus \\ R(-3,-2)^2 \\ \oplus \\ R(-4,-1)^2 \end{matrix} \leftarrow R(-4,-2)^2 \leftarrow 0$
4	$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{matrix} R(-2,-3) \\ \oplus \\ R(-3,-1) \\ \oplus \\ R(-3,-2)^2 \\ \oplus \\ R(-4,-2) \\ \oplus \\ R(-5,-1) \end{matrix} \leftarrow \begin{matrix} R(-3,-3) \\ \oplus \\ R(-4,-3) \\ \oplus \\ R(-5,-2)^2 \end{matrix} \leftarrow R(-5,-3) \leftarrow 0$

The six Betti types - continued

Type	Bigraded Minimal Free Resolution of I
3	$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{matrix} R(-2,-4) \\ \oplus \\ R(-3,-1) \\ \oplus \\ R(-3,-2)^2 \\ \oplus \\ R(-3,-3) \\ \oplus \\ R(-4,-2) \\ \oplus \\ R(-5,-1) \end{matrix} \leftarrow \begin{matrix} R(-3,-4)^2 \\ \oplus \\ R(-4,-3)^2 \\ \oplus \\ R(-5,-2)^2 \end{matrix} \leftarrow \begin{matrix} R(-4,-4) \\ \oplus \\ R(-5,-3) \end{matrix} \leftarrow 0$
2	$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{matrix} R(-2,-3) \\ \oplus \\ R(-3,-2)^4 \\ \oplus \\ R(-4,-1)^2 \end{matrix} \leftarrow \begin{matrix} R(-3,-3)^2 \\ \oplus \\ R(-4,-2)^3 \end{matrix} \leftarrow R(-4,-3) \leftarrow 0$
1	$0 \leftarrow I \leftarrow R(-2,-1)^4 \leftarrow \begin{matrix} R(-2,-4) \\ \oplus \\ R(-3,-2)^4 \\ \oplus \\ R(-4,-1)^2 \end{matrix} \leftarrow \begin{matrix} R(-3,-4)^2 \\ \oplus \\ R(-4,-2)^3 \end{matrix} \leftarrow R(-4,-4) \leftarrow 0$

Table: The six Betti types for bidegree (2,1) four-generated ideals

Relations between syzygies and singularity types

Type	Lin. Syz.	Emb. Pri.	Sing. Loc.	Example
1	none	$\mathfrak{m}$	T	$(s^2u+stv, t^2u, s^2v+stu, t^2v+stv)$
2	none	$\mathfrak{m}, P_1$	$C \cup L_1$	$(s^2u, t^2u, s^2v + stu, t^2v + stv)$
3	1 type $(1,0)$	$\mathfrak{m}$	L_1	$(s^2u + stv, t^2u, s^2v, t^2v + stu)$
4	1 type $(1,0)$	$\mathfrak{m}, P_1$	L_1	$(stv, t^2v, s^2v - t^2u, s^2u)$
5a	1 type $(0,1)$	P_1, P_2	$L_1 \cup L_2 \cup L_3$	$(s^2u, s^2v, t^2u, t^2v + stv)$
5b	1 type $(0,1)$	P_1	$L_1 \cup L_2$	$(s^2u, s^2v, t^2u, t^2v + stu)$
6	2 type $(0,1)$	none	$\emptyset$	(s^2u, s^2v, t^2u, t^2v)

Table: The primary decomposition and singularities for the six Betti types

- T = twisted cubic curve, C = smooth plane conic L_i = lines
- $\mathfrak{m} = \langle s, t, u, v \rangle$, $P_i = \langle l_i, s, t \rangle$, l_i = linear form of bidegree $(0,1)$