Syzygy Theorems via Comparison of Order Ideals

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Evans-Griffith Syzygy Theorem

The Evans-Griffith Syzygy Theorem (1981) asserts:

Theorem (Evans-Griffith)

A non-free, finitely generated and finite projective dimension k^{th} module of syzygies over a Cohen-Macaulay local ring R containing a field, has rank at least k.

It is known

- for rings R containing a field
- for graded resolutions over $R = V[[x_1, \ldots, x_n]]$, V a DVR
- for R of any (including mixed) characteristic and $\dim R \leq 5$.

Order Ideals

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If E is the kernel of

$$E \longrightarrow R^m \xrightarrow{\begin{bmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{bmatrix}} R^n$$

and if $e_1 \in E$

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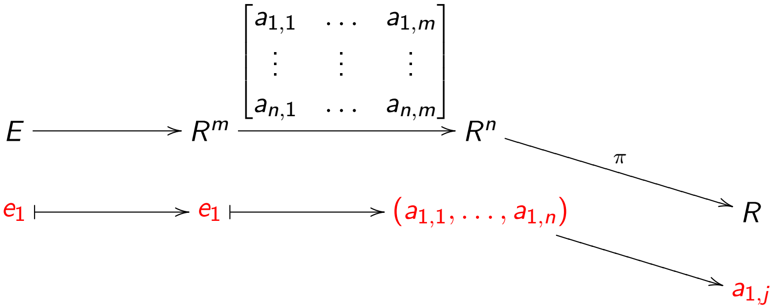
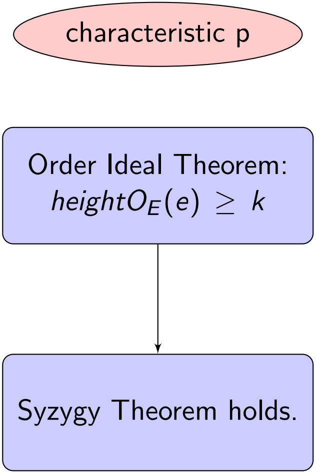
Then $(a_{1,1}, \ldots, a_{1,n}) \subseteq O_E(e_1)$:



Connection between order ideals and ranks of syzygies



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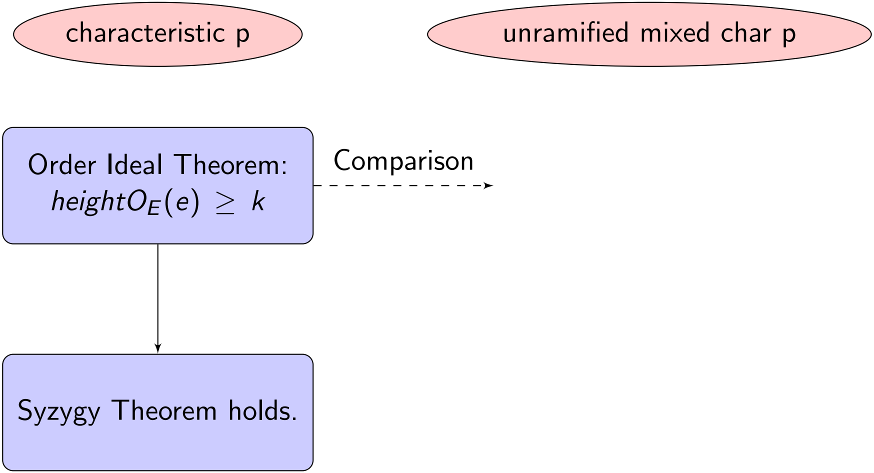

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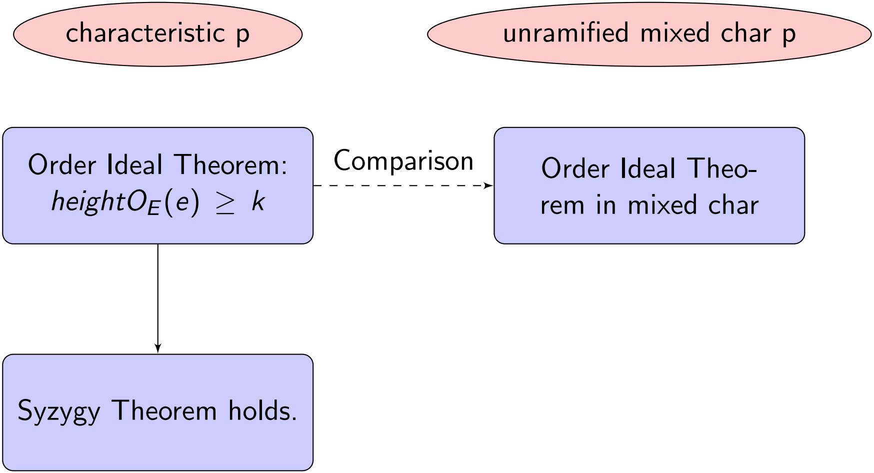

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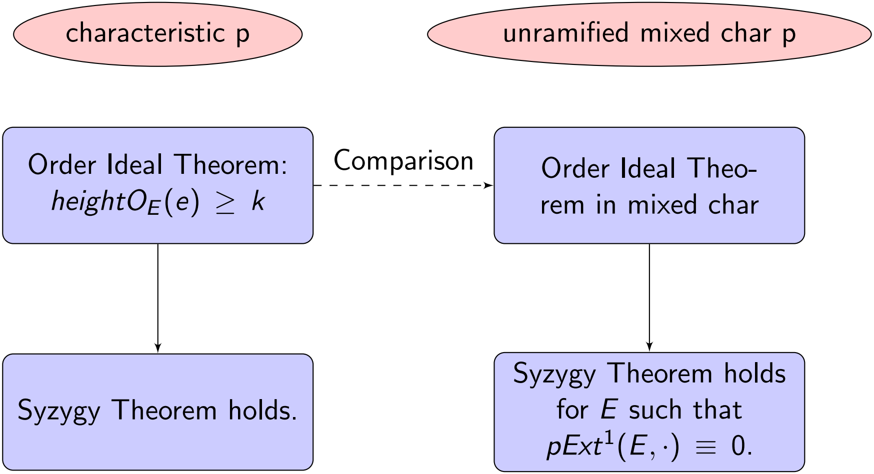

Superficial elements

Definition (Samuel)

Let R be a ring , I an ideal, M an R-module. We say a non-zerodivisor $x \in I$ is a superficial element of I with respect to M if

$$(I^{n+1}M :_M x) = I^n M$$

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(For the purposes of this talk one may just think $x \in m - m^2$.)

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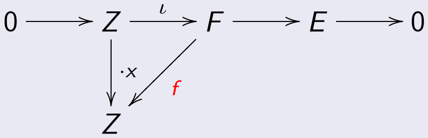
Then one can build a diagram



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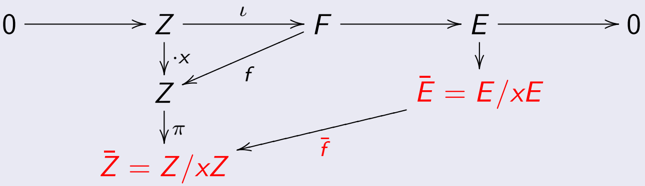

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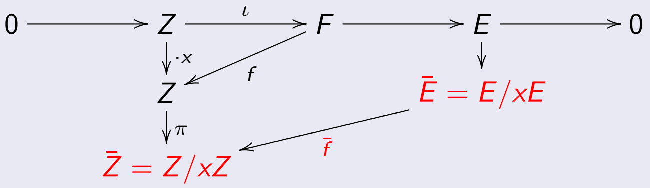

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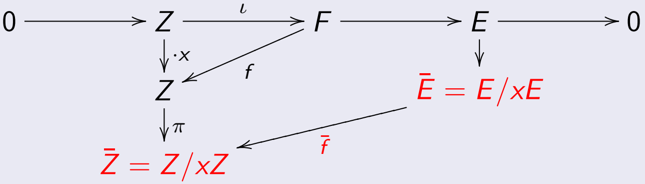

such that $Im(\bar{f}) \not\subseteq m\bar{Z}$ and consequently $htO_E(e) \geq htO_{\bar{Z}}(\bar{f}(\bar{e}))$.

Main Consequences on the Comparison Theorem

To apply the Comparison Theorem with $x = p$ we need that p be unramified i.e. $p \in m - m^2$.

Theorem (Griffith, -)

Let (R, m) be an unramified Cohen-Macaulay local ring of mixed characteristic. Assume that M is a finite projective dimension R-module such that for a fixed k, $pExt_R^{k+1}(M, \cdot) \equiv 0$. Then the Syzygy theorem holds for every j^{th} syzygy of M with $j \geq k$.

Main Consequences on the Comparison Theorem

Theorem (Griffith, -)

Let (R, m) be an unramified Cohen-Macaulay local ring of mixed characteristic. Then the Syzygy Theorem holds over R for syzygies of modules of the type R/Q with $Q \in Spec(R)$.

The Strong Syzygy Theorem

Theorem (Strong Syzygy Theorems)

Let (R, m) be a local ring of unmixed ramified characteristic p and M an R-module that satisfies any of the hypotheses

- *M is annihilated by p (Shamash)*
- *M viewed as $R/(p)$-module is weakly liftable to R (ADS)*
- *$\bar{M} = M/xM \simeq (0 :_M x)$ i.e. there is a four-term exact sequence $0 \to \bar{M} \xrightarrow{\delta} \bar{F} \to \bar{E} \to \bar{M} \to 0$*

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. Then

1. *there is a short exact sequence* $0 \to Syz_{k-1}(\bar{M}) \to \overline{Syz_k(M)} \to Syz_k(\bar{M}) \to 0$
2. $\operatorname{rank} Syz_k(M) \geq 2k - 1$ *for* $1 \leq k \leq pd(M) - 3$
3. $\beta_k^R(M) = \beta_{k-1}^{\bar{R}}(M) + \beta_k^{\bar{R}}(M)$ *for* $2 \leq k \leq pdM - 1$,

Applications to Weak Lifting

Definition

An $\bar{R}$-module N is weakly liftable to R if it is a direct summand of a module $M \otimes \bar{R}$ such that $Tor_i^R(M, R/(p)) = 0$ for $i \geq 1$.

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Find interesting classes of examples !

Thank You

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