The complexity of bounding projective dimension

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Notation

Joint work with C. Huneke, P. Mantero and J. McCullough [HMMS].

$R = K[X_1, \ldots, X_N]$ polynomial ring
$I = (f_1, \ldots, f_n)$ homogeneous ideal of R

N= very large integer (assumed unknown)	number of variables of R
n = (smaller) known integer	number of generators of I
$d_1, \ldots, d_n$ = positive integers	degrees of generators of I

(often $d_1 = \ldots = d_n = d$)

Stillman's Question

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- Hilbert's Syzygy Theorem guarantees $pd(S/I) \leq N$, but we seek a bound independent of N.
- This question is still open in full generality!

Approaches

Two ideas in pursuing this question:

1. relevant "variable" counting if I is contained in a K-subalgebra of R generated by a regular sequence $y_1, \ldots, y_s$, then

$$pd(R/I) = pd(K[y_1, \ldots, y_s]/I) \leq s$$

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2. case analysis based on fixing the multiplicity of I

   Eisenbud-Huneke I generated by 3 quadrics has $pd(R/I) \leq 4$
   Engheta I generated by 3 cubics has $pd(R/I) \leq 36$
   HMMS I generated by 4 quadrics has $pd(R/I) \leq 9$

The multiplicity approach

Given I of height h and generated in degree d, one has $e(R/I) \leq d^h$.

Inspired by Stillman's question we asked:

Question (MS)

Can one bound the projective dimension of all (unmixed) ideals of a given height h and multiplicity e?

The case $e = 2, h = 2$

Engheta Let J be a height two unmixed ideal of multiplicity two. Then J is one of the following:

1. $(x, y) \cap (w, z)$ with independent linear forms w, x, y, z.
2. (x, yz) with independent linear forms x, y, z.
3. (x, q) a prime ideal, with x linear form and q irreducible quadratic.
4. (x, y^2) with independent linear forms x, y.
5. $(x, y)^2 + (ax + by)$ with independent linear forms x, y and a, b such that x, y, a, b form a regular sequence.

- Note: by relevant "variable" counting $pd\, R/J \leq 4$.

The answer is NO (even under additional assumptions)

Theorem (HMMS)

Let K be an algebraically closed field.

For any integers $h, e \geq 2$ with $(h, e) \neq (2, 2)$ and for any integer $p \geq 5$, there exists an ideal $I_{h,e,p}$ in a polynomial ring over K such that

- $I_{h,e,p}$ *is a primary ideal*
- $I_{h,e,p}$ *has height h and multiplicity e*
- $\sqrt{I_{h,e,p}}$ *is a linear prime*
- $pd(R/I_{h,e,p}) \geq p$.

We call such ideals multiple structures on the corresponding linear space.

Classifying multiple structures on linear spaces

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- Manolache classifies locally Cohen-Macaulay multiple structures in codimension 2 and multiplicity at most 4
- Manolache classifies locally complete intersection multiple structures of degree at most 6
- we (HMMS) show that no such finite characterization of multiple structures is possible if one only assumes Serre's (S_1) property holds.

Relation to vector bundles

- Hartshorne's Conjecture in Codimension Two
  Any smooth variety of codimension two in a projective space $\mathbb{P}^N$ of dimension $N \geq 6$ is a complete intersection.

- Rank Two Bundle Conjecture
  Any vector bundle of rank two on $\mathbb{P}^N$, $N \geq 6$ splits.

- Vatne - Conjecture on Triple Linear Schemes

  Consider a linear subspace L of dimension $N \geq 6$ and a (locally) CohenMacaulay scheme X, $e(X) = 3$ supported on L.

  Then there exists a (locally) CohenMacaulay scheme Y, $e(Y) = 2$ supported on L such that $Y \subset X$.

Idea of proof

We wish to construct $I_{h,e,p}$ of height h, multiplicity e and $pd(R/I) \geq p$.

- only need $I_{2,e,p}$ and $I_{3,3,p}$ as $I_{h+1,e,p+1} = I_{h,e,p} + (y)$
- produce an ideal L of appropriate height using a 3-generated ideal with large projective dimension (many constructions known)
- use linkage appropriately to obtain $I_{2,e,p}$ or $I_{3,3,p}$ from L

Details

Suppose $f, g, h \in R_d$, $\mathrm{ht}(x, y, f, g, h) \geq 4$ and $pd\, R/(f, g, h) = p$. Let

$$L = (x, y)^3 + (y^2 f + xyg + x^2 h).$$

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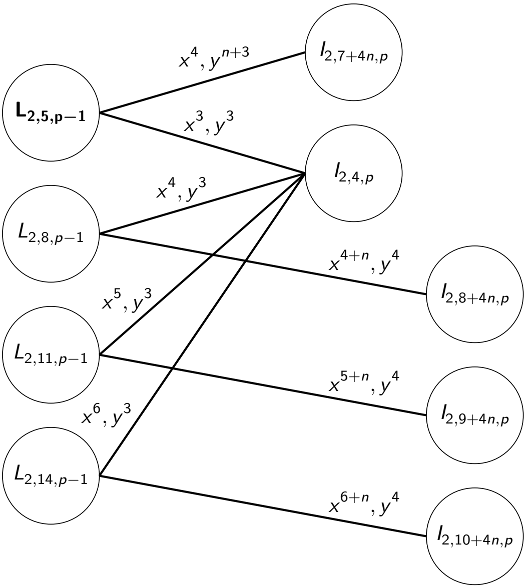
Then R/L has the following free resolution:

$$0 \to R \xrightarrow{d_3 = \begin{pmatrix} f & g & h & -y & x \end{pmatrix}^T} R^5 \xrightarrow{d_2} R^5 \xrightarrow{d_1} R \to R/L \to 0,$$

Dualize and check that $pd\, Ext^2(R/L, R) = p - 1$.

Set $I = C : L$, where C is a complete intersection contained in L.
Then $Ext^2_R(R/L, R) = I/C$, so $pd\, R/I = p$.

Linkage



$\mathbf{L_{2,5,p-1}}$
$L_{2,8,p-1}$
$L_{2,11,p-1}$
$L_{2,14,p-1}$
$I_{2,7+4n,p}$
$I_{2,4,p}$
$I_{2,8+4n,p}$
$I_{2,9+4n,p}$
$I_{2,10+4n,p}$
x^4, y^{n+3}
x^3, y^3
x^4, y^3
x^5, y^3
x^6, y^3
x^{4+n}, y^4
x^{5+n}, y^4
x^{6+n}, y^4