Syzygies and singularities of tensor product surfaces

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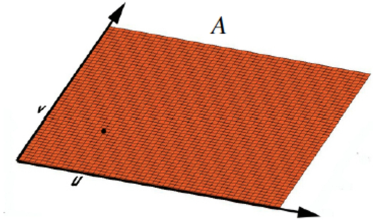
Surface modeling

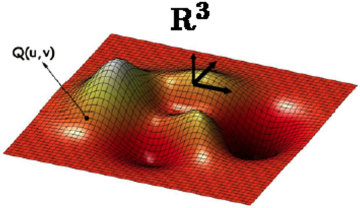
In Computer Aided Geometric Design

surface splines are made from patches defined parametrically by rational maps

$$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\phi(x, y) = \left(\frac{p_1(x, y)}{p_0(x, y)}, \frac{p_2(x, y)}{p_0(x, y)}, \frac{p_3(x, y)}{p_0(x, y)} \right)$$





Tensor product surface

Instead of $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ or $\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^3$ (triangular surface), a **tensor product surface** is the image of a **bi-homogeneous parametrization** map

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$

$$[(s:t),(u:v)] \mapsto [p_0(s,t,u,v) : p_1(s,t,u,v) : p_2(s,t,u,v) : p_3(s,t,u,v)]$$

with

$$\deg(\mathbf{s}) = \deg(\mathbf{t}) = (\mathbf{1},\mathbf{0}) \text{ and } \deg(\mathbf{u}) = \deg(\mathbf{v}) = (\mathbf{0},\mathbf{1}).$$

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- This is a special case of toric parametrization.
- We restrict to the case of a bidegree $(2,1)$ parametrization (yields a surface ruled by lines and quadrics) w/o base locus.

Example

$$\phi : [(s : t), (u : v)] \longmapsto [s^2u : s^2v : t^2u : t^2v + stv]$$

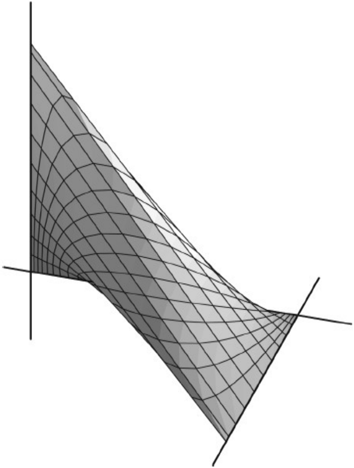

Figure: Three double lines on a bidegree $(2, 1)$ surface

Example - continued

Let $I = (s^2u, s^2v, t^2u, t^2v + stv)$ be the parametrization ideal.

- I has a **linear syzygy** of bidegree (0, 1)

$$v(s^2u) - u(s^2v) = 0$$

- $X = Im(\phi)$ has the **implicit equation**:

$$X = \mathbf{V}(x_0x_1^2x_2 - x_1^2x_2^2 + 2x_0x_1x_2x_3 - x_0^2x_3^2).$$

- the reduced codimension one **singular locus** of X is:

$$\mathbf{V}(x_0, x_2) \cup \mathbf{V}(x_1, x_3) \cup \mathbf{V}(x_0, x_1).$$

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Linear syzygies & the Segre-Veronese $\Sigma_{2,1}$

Proposition (Schenck-S.-Validashti)

The ideal $I = (p_0, p_1, p_2, p_3)$

1. *has a unique linear syzygy of bidegree* $(0,1)$ *iff* $\mathbb{P}\langle p_0, p_1, p_2, p_3\rangle \cap \Sigma_{2,1}$*,contains a* $\mathbb{P}^1$ *fiber of* $\Sigma_{2,1}$.
   e.g. $Span\langle s^2u, s^2v\rangle$ *is a* $\mathbb{P}^1$ *fiber*
2. *has a pair of linear syzygies of bidegree* $(0,1)$ *iff* $\mathbb{P}\langle p_0, p_1, p_2, p_3\rangle \cap \Sigma_{2,1} = \Sigma_{1,1}$.
3. *has a unique linear syzygy of bidegree* $(1,0)$ *iff* $\mathbb{P}\langle p_0, p_1, p_2, p_3\rangle \cap Q$ *contains a* $\mathbb{P}^1$ *fiber of* Q.

Main result

Theorem (Schenck-S.-Validashti)

There are exactly 6 (families of) resolutions for ideals generated by four bidegree (2,1) forms without basepoints.

This uses

- the geometry of the Segre-Veronese variety (to determine how many linear and quadratic syzygies may exist)
- the Buchsbaum-Eisenbud exactness criterion (to write down an explicit resolution when linear syzygies exist)
- bigraded gins to determine the resolution in the generic case

Implicitization via the approximation complex

- Sederberg and Chen (1995) introduced for implicitization purposes a method termed as **moving curves** and surfaces.
- Cox realized they were using syzygies with several coauthors (Busé, Chen, D'Andrea, Goldman, Sederberg, Zhang).
- Jouanolou and Busé (2002) gave a sound theoretical basis for the method of Sederberg-Chen via **approximation complexes**, a tool in homological algebra developed by Herzog–Simis–Vasconcelos.

Implicitization via the approximation complex

- **Step 1:** Find the syzygies on p_0, p_1, p_2, p_3
- **Step 2:** Represent them as **linear combinations**

$$L_j = \alpha_0^{(j)} x_0 + \alpha_1^{(j)} x_1 + \alpha_2^{(j)} x_2 + \alpha_3^{(j)} x_3$$

- **Step 3:** Rewrite the syzygies of degree ν in terms of a monomial basis $\{m_\beta\}_{|\beta|=\nu}$ of $k[s,t,u,v]_\nu$

$$L_j = \sum_{i=0}^{3} \sum_{|\beta|=\nu} c_{i,\beta}^{(j)} m_\beta x_i = \sum_{|\beta|=\nu} \left(\sum_{i=0}^{3} c_{i,\beta}^{(j)} x_i \right) m_\beta$$

- **Step 4:** The implicit equation is the **gcd of the maximal minors** of the matrix $M = (\sum_{i=0}^{3} c_{i,\beta}^{(j)} x_i)_{\beta,j}$ for well-chosen ν.

Implicitization example: $\nu = (1,1)$

$$vx_0 - ux_1 = 0$$

$$\begin{aligned} s^2x_2 - t^2x_0 &= 0 \\ s^2x_3 - (st+t^2)x_1 &= 0 \\ tux_3 - (sv+tv)x_2 &= 0 \\ sux_3 - svx_2 - tvx_0 &= 0 \end{aligned}$$

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & x_2 & x_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -x_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -x_0 & -x_1 & \cdot & \cdot \\ x_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x_3 \\ -x_0 & \cdot & \cdot & \cdot & \cdot & \cdot & -x_2 & -x_2 \\ \cdot & x_1 & \cdot & \cdot & \cdot & \cdot & x_3 & \cdot \\ \cdot & -x_0 & \cdot & \cdot & \cdot & \cdot & -x_2 & -x_0 \\ \cdot & \cdot & x_1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -x_0 & x_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -x_0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Theorem (Schenck-S.-Validashti)

In fact the implicit equation is itself a smaller minor !

Relations between syzygies and singularity types

Type	Lin. Syz.	Emb. Pri.	Sing. Loc.	Example
1	none	$\mathfrak{m}$	T	$(s^2u+stv, t^2u, s^2v+stu, t^2v+stv)$
2	none	$\mathfrak{m}, P_1$	$C \cup L_1$	$(s^2u, t^2u, s^2v+stu, t^2v+stv)$
3	1 type $(1,0)$	$\mathfrak{m}$	L_1	$(s^2u+stv, t^2u, s^2v, t^2v+stu)$
4	1 type $(1,0)$	$\mathfrak{m}, P_1$	L_1	$(stv, t^2v, s^2v-t^2u, s^2u)$
5a	1 type $(0,1)$	P_1, P_2	$L_1 \cup L_2 \cup L_3$	$(s^2u, s^2v, t^2u, t^2v+stv)$
5b	1 type $(0,1)$	P_1	$L_1 \cup L_2$	$(s^2u, s^2v, t^2u, t^2v+stu)$
6	2 type $(0,1)$	none	$\emptyset$	(s^2u, s^2v, t^2u, t^2v)

Table: The primary decomposition and singularities for the six Betti types

- T = twisted cubic curve, C = smooth plane conic L_i = lines
- $\mathfrak{m} = \langle s,t,u,v\rangle$, $P_i = \langle l_i, s, t\rangle$, l_i = linear form of bidegree $(0,1)$