Resolutions for powers of ideals and applications to symbolic powers

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Outline

The goal for this talk is:

For homogeneous ideals I, understand

- the minimal free resolution for the powers of I i.e. I^m
- whether there is equality between symbolic and ordinary powers

$$I^{(m)} = I^m$$

Motivation I: geometry

The m-th **symbolic power** $I^{(m)}$ of a radical ideal $I \subset R$ is

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- Zariski-Nagata: if P is prime and $\mathbf{X} = V(P)$ (algebraic variety), then $P^{(m)}$ = the set of forms that vanish to order at least m at every point of $\mathbf{X}$

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- in characteristic 0, $P^{(m)}$ = the forms that vanish together with their first $m-1$ partial derivatives at every point of $\mathbf{X}$.

Motivation I: geometry

Certain classes of varieties $\mathbf{X}$ have $I_{\mathbf{X}}^{(m)} = I_{\mathbf{X}}^m$ for all $m > 0$:

- $I_{\mathbf{X}}$ = complete intersection
- $I_{\mathbf{X}}$ = maximal minors of a generic matrix of variables (DeConcini, Eisenbud, Processi)

Our goal: add to this list.

Motivation II: combinatorics

Classes of ideals I from combinatorics that have $I^{(m)} = I^m, \forall n > 0$:

- $I =$ edge ideal of a bipartite graph (Simis-Vasconcelos-Villarreal, Sullivant)
- $I =$ edge (face) ideal of a hypergraph with the MFMC property (Gitler-Valencia-Villarreal)
- the packing problem states that a squarefree monomial ideal I has $I^{(n)} = I^n, \forall n > 0$ iff I is packed, i.e. upon replacing any variables by 0 or 1 the resulting ideal I' contains a regular sequence of $\mathrm{ht}(I')$ monomials (Conforti-Conjuelos)

Definition

A subscheme $\mathbf{X}$ is *locally a complete intersection* (l.c.i) if the localization of $I_{\mathbf{X}}$ at any prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \neq \mathfrak{m}$ and $I_{\mathbf{X}} \subseteq \mathfrak{p}$ is a complete intersection.

Examples:

- complete intersections
- points in $\mathbb{P}^N$,
- points in $\mathbb{P}^1 \times \mathbb{P}^1$

Useful fact: if I is l.c.i then $I^{(m)} = (I^m)^{\text{sat}}$.

Main results

Theorem

Let $I \subset R = k[x_0, \ldots, x_n]$ be a homogeneous perfect (ACM) locally complete intersection (l.c.i) ideal of codimension two with resolution

$$0 \to F \longrightarrow G \to I \to 0.$$

If $\min\{\mu(I) - 1,\ m\} \leq n$*, then* $Sym^m I \cong I^m$ *and the graded minimal free resolution of* I^m *is*

$$0 \to \bigwedge^m F \to \bigwedge^{m-1} F \otimes_R Sym^1 G \to \bigwedge^{m-2} F \otimes_R Sym^2 G \to \cdots$$

$$\cdots \to \bigwedge^2 F \otimes_R Sym^{m-2} G \to F \otimes_R Sym^{m-1} G \to Sym^m G \twoheadrightarrow I^m.$$

Betti numbers

Suppose

- $I \subset R = k[x_0, \ldots, x_n]$ with $\mu(I) = r$.
- I is a perfect codimension 2 l.c.i

If $\min\{r-1, m\} \leq n$ then

$$\beta_i(R/I^m) = \operatorname{rank} \bigwedge^{i-1}(R^{r-1}) \otimes_R \operatorname{Sym}^{m-i+1}(R^r) = \binom{r-1}{i-1}\binom{r+m-i}{m-i+1}$$

$$pd(R/I^m) = \min\{r, m+1\}$$

$$I^m \text{ is saturated } \Leftrightarrow \min\{r, m+1\} \leq n$$

Equality of symbolic and ordinary powers

Theorem

Let $I = I_X$ be the saturated homogeneous ideal defining a subscheme $X \subset \mathbb{P}^n$ such that

- *codim*$(X) = 2$;
- X *is arithmetically Cohen-Macaulay;*
- X *is a locally complete intersection.*

Then the following conditions are equivalent:

(a) $I^{(n)} = I^n$;

(b) $I^{(m)} = I^m$ *for all* $m \geq 1$;

(c) I *has at most* n *minimal generators.*

Furthermore, if $m < n$, *then* $I_X^{(m)} = I_X^m$ *regardless of the number of generators.*

Equality of symbolic and ordinary powers

Corollary

Let $I = I_X$ be the saturated defining ideal of an arithmetically Cohen-Macaulay set of points in $\mathbb{P}^1 \times \mathbb{P}^1$. Then

- $I^{(2)} = I^2$ *always*
- $I^{(3)} = I^3$ *if and only if* I *is a complete intersection or* I *is an almost complete intersection (i.e., it has exactly three minimal generators).*
- $I^{(m)} = I^m$ *for all* $m \geq 0$ *if and only if* $I^{(3)} = I^3$.

A question of Huneke

Question (Huneke)

Given and ideal I, is there N such that if $I^{(m)} = I^m$ for all $m \leq N$, then one can conclude that $I^{(m)} = I^m$ for all $m > 0$?

Partial answer:

If I is a perfect codimension two l.c.i. defining a scheme in $\mathbb{P}^d$, then $N = d$ works. In fact, if $I^{(d)} = I^d$ then $I^{(m)} = I^m$ for all $m > 0$?

A question of Roemer

Question (Römer)

Let I be a homogeneous ideal of $R = k[x_0, \ldots, x_n]$. Does the following bound hold for all $i = 1, \ldots, p$:

$$\beta_i(R/I) \leq \frac{1}{(i-1)!(p-i)!} \prod_{j \neq i} M_j$$

where $M_i := \max\{j \mid \beta_{i,j}(R/I) \neq 0\}$?

Partial answer:

The bound above holds for powers of perfect codimension two l.c.i's.