Symbolic versus regular powers of ideals of points

Alexandra Seceleanu
(joint with Brian Harbourne)

October 2013
AMS Meeting, Louisville KY

Symbolic vs regular powers of homogeneous ideals

Theorem

For any homogeneous ideal $I \subseteq K[\mathbb{P}^N] = K[x_0, \ldots, x_N]$*, the following containment holds*

$$I^{(Nr)} \subseteq I^r, \forall r \geq 1$$

proven by

- Ein-Lazarsfeld-Smith (2001), for I unmixed, using multiplier ideals
- Hochster-Huneke (2002) using tight closure methods (reduction to char p)

Improving the containments

The theorem states that $I^{(Nr)} \subseteq I^r, \forall r \geq 1$.

- Is it possible to replace Nr in the symbolic exponent by another linear function of r, say cr (with $c \leq N$) while preserving the containment?
  - Bocci-Harbourne (2010) showed that the smallest such c is $c = N$, that is the containment cannot be strengthened in this way

- Is it possible to decrease the symbolic exponent Nr replacing it by a $Nr - c$ for some constant c?
  - i.e. is there a c such that the containment $I^{(Nr-c)} \subseteq I^r$ holds $\forall r \geq 1$ at least for some classes of homogeneous ideals I?

A question and a conjecture

Question (Huneke)

Does

$$I^{(2\cdot 2-1)} = I^{(3)} \subseteq I^2$$

always holds in the case of $I \subseteq K[\mathbb{P}^2]$ *defining a reduced set of points of* $\mathbb{P}^2$*?*

Conjecture (Harbourne)

In the case of $I \subseteq K[\mathbb{P}^N]$ *defining a reduced set of points of* $\mathbb{P}^N$

$$I^{(Nr-N+1)} \subseteq I^r$$

holds for all $r \geq 1$ *and all* $N \geq 1$.

First counterexamples to $I^{(3)} \subseteq I^2$

Dumnicki, Szemberg and Tutaj-Gasińska (2013) consider

$$I = (x_0(x_1^3 - x_2^3), x_1(x_0^3 - x_2^3), x_2(x_0^3 - x_1^3))$$

- is the ideal of 12 points arising as pairwise intersections of 9 lines
- each point lies on 3 lines and each line passes through 4 points
- this point and line configuration is dual to the Hesse configuration

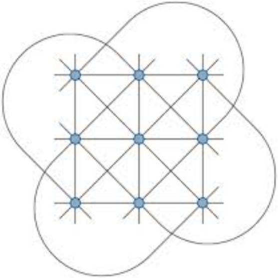

Figure: The Hesse configuration

Positive characteristic counterexamples to $I^{(3)} \subseteq I^2$

Harbourne (2013)

- any 12 of the 13 $\mathbb{Z}/3\mathbb{Z}$-points in $\mathbb{P}^2$ over any field K of characteristic 3

$$I = \left(x_0x_1(x_0^2 - x_1^2), x_0x_2(x_0^2 - x_2^2), x_1x_2(x_1^2 - x_2^2), x_0(x_0^4 - x_1^4 + x_1^2x_2^2 - x_2^4)\right)$$

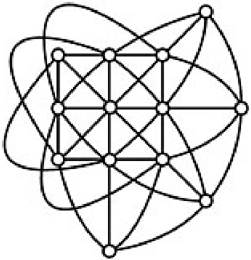

Figure: The incidence structure of $\mathbb{P}^2_{\mathbb{F}_3}$

New counterexamples to $I^{(Nr-(N-1))} \subseteq I^r$

Theorem (Harbourne-S.)

Let K be a field of characteristic $p > 0$ and let K' be the subfield of order p.

Let $I \subseteq K[\mathbb{P}^N] = K[x_0, \ldots, x_N]$ be the ideal of all of the K'-points of $\mathbb{P}^N_K$ but one.

We prove that $I^{(Nr-(N-1))} \not\subseteq I^r$ always holds for the following cases:

1. $p > 2$, $r = 2$ *and* $N = (p+1)/2$
2. $r = (p+N-1)/N$, $p > (N-1)^2$ *and* $p \equiv 1 (\bmod N)$.

Proof ideas

Let $I \subseteq K[\mathbb{P}^N] = K[x_0, \ldots, x_N]$ be the ideal of all K' points of $\mathbb{P}^N_K$ but one.

- we show that the smallest degree n such that I_n contains a form which does not vanish at every point of $\mathbb{P}^N_K$ is $n = N(p-1)+1$.
- hence the smallest degree n such that I^r_n contains a form which does not vanish at every point of $\mathbb{P}^N_K$ is $n = r\,[N(p-1)+1]$.
- we construct (explicitly) a form of degree $r\,[N(p-1)+1]-1$ in $I^{(rN-N+1)}$.

Relation to basic double links and Cayley-Bacharach

To determine the degrees of the minimal generators of the ideal I of all K' points of $\mathbb{P}_K^N$ but $q = [1 : 0 : \ldots : 0]$ we show

- $I = J + (x_0)B$, where
  - $J \subset K[x_1, \ldots, x_N]$ is the ideal of all points in $\mathbb{P}_K^{N-1}$
  - B is be the ideal of the $p^N - 1$ points in $\mathbb{P}_K^N$ which are not on $x_0 = 0$ and are distinct from q hence I is a basic double link of B.

$$I = \underbrace{\left(x_1x_2(x_1^2 - x_2^2)\right)}_{J} + (x_0)\underbrace{\left(x_1(x_0^2 - x_1^2), x_2(x_0^2 - x_2^2), x_0^4 - x_1^4 + x_1^2x_2^2 - x_2^4\right)}_{B}$$

- B is a complete intersection C (defining the finite affine N-space over K) except a point, so the Cayley Bacharach Theorem yields the smallest degree of a form in $B \setminus C$

Open questions

1. Are the counterexamples for $r > 2$ or for $N > 2$ purely a positive characteristic phenomenon?

2. Is it true that $I^{(Nr-1)} \subseteq I^r$ holds for all radical ideals I of finite sets of points in $\mathbb{P}^N$ for all $r \geq 1$ as long as $N > 2$?

3. Revised version of Huneke's question:

   Is it always true for the ideal I of a finite set of points in $\mathbb{P}^3$ that $I^{(5)} \subseteq I^2$?