Polynomial Betti numbers

Alexandra Seceleanu
(with L. Avramov and Z. Yang)

University of Nebraska-Lincoln

The problem

Setup:

- R – local (or graded) ring
- M – R-module
- $\beta_i^R(M)$ – i^{th} total Betti number of M
- $P_M^R(t) = \sum \beta_i^R(M)t^i$ – Poincaré series of M

Phylosophy:

- the first few Betti numbers of M reflect properties of M
- the long-term behavior of Betti numbers of R-modules reflects the properties of R

Some classical examples

The long-term behavior of Betti numbers of R-modules reflects the properties of R:

- Auslander-Buchsbaum-Serre:
  $\beta_i^R(M) = 0, \forall M, i \gg 0 \iff R$ is regular
- Eisenbud:
  $\beta_i^R(M) = \mathrm{constant}, \forall \mathrm{M}, \mathrm{i} \gg 0 \iff \mathrm{R}$ is a hypersurface

The question

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Equivalently

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What rings R have the property that for each module M there is a polynomial $p_M(t)$ and $c_M \in \mathbb{N}$ such that $P_M^R(t) = \frac{p_M(t)}{(1-t)^{c_M}}$.

We say R has polynomial Betti numbers if it is as above.

Polynomial growth for Betti numbers

Gulliksen:

Betti numbers over R are bounded by polynomials $\implies$ R is a CI.

In this case $\beta^R_{2i}(M)$ and $\beta^R_{2i+1}(M)$ are given by two polynomials.

Example

$R = k[x,y]/(x^3,y^3), M = R/(x,y)^2$

$$\beta^R_i(M) = \begin{cases} \frac{3}{2}i+1, & \text{if i is even} \\ \frac{3}{2}i+\frac{3}{2}, & \text{if i is odd} \end{cases}$$

More about Poincaré series over CI's

- If R is a CI, then $Ext^{\bullet}_R(M,k)$ is a $k[\chi_1,\dots,\chi_c]$-module with
  - $\deg(\chi_i) = 2$
  - $c = \operatorname{codim}(R)$
- Thus
$$P^R_M(t) = H\left(Ext^{\bullet}_R(M,k)\right) = \frac{p_M(t)}{(1-t^2)^c}$$

If R has polynomial Betti numbers, then $(1+t)^c \mid p_M(t)$ for all M.

Main result – I

Let $R = P/(f_1, f_2, \ldots, f_c)$ be a CI $\qquad (c = \mathrm{codim R})$.

Avramov If R has polynomial Betti numbers, then $f_1, \ldots, f_{c-1} \in m^2 \setminus m^3$.

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Theorem (Avramov-S.-Yang: Sufficient conditions)

(i) *Assume R is graded. Then R has polynomial Betti numbers if*

$$\deg(f_1) = \deg(f_2) = \ldots = \deg(f_{c-1}) = 2$$

(ii) *Assume R is local. Then R has polynomial Betti numbers if*

$$f_1, f_2, \ldots, f_{c-1} \in m^2 \setminus m^3$$

and their images in $\mathrm{gr}(P)$

$$f_1^*, f_2^*, \ldots, f_{c-1}^* \in \frac{m^2}{m^3} \text{ form a regular sequence.}$$

Sufficient conditions sketch

- Let $R = P/(f_1, f_2, \ldots, f_c)$ be a CI and M an R-module
- The hypothesis $e(P/(f_1, f_2, \ldots, f_{c-1})) = 2^{c-1}$
- We can adjust to having $R = P/(g_1, g_2, \ldots, g_c)$ such that
  - $Q = P/(g_1, g_2, \ldots, g_{c-1})$ has $e(Q) = 2^{c-1}$ (min. mult.) and
  - $P_M^R(t) = \frac{P_M^Q(t)}{(1-t^2)}$ + a polynomial (χ_C almost regular on $Ext_R^\bullet(M, k)$).
- $P_M^Q(t) = \frac{p_Q(t)}{(1-t)^{c-1}}$ with $(1+t) \mid p_Q(t)$ (Avramov, Şega)
- Now $P_M^R(t) = \frac{P_M^Q(t)}{(1-t^2)} = \frac{p_Q(t)}{(1-t)^c(1+t)} = \frac{p_M(t)}{(1-t)^c}$.

Main result - II

Let $R = P/(f_1, f_2, \ldots, f_c)$ be a CI $\qquad (c = \mathrm{codimR})$

Theorem (Avramov-S.-Yang: Necessary & sufficient conditions)

(i) *Assume R is graded. Then R has polynomial Betti numbers iff*

$$\deg(f_1) = \deg(f_2) = \ldots = \deg(f_{c-1}) = 2$$

(ii) *Assume R is local and $c \leq 4$. Then R has polynomial Betti numbers iff $f_1, f_2, \ldots, f_{c-1} \in m^2 \setminus m^3$ and the initial forms $f_1^*, f_2^*, \ldots, f_{c-1}^* \in \mathrm{gr}(P)$ form a regular sequence.*

Necessary conditions sketch

When R doesn't satisfy our conditions, we produce explicit modules without polynomial Betti numbers: $c \leq 4$, $I_2^* = (f_1^*, f_2^*, f_3^*) \subset \mathrm{gr}(P)$

- if $I_2^* \subset (\ell_1, \ell_2)$ then set $M = R/(m^2 + (\ell_1, \ell_2))$
- if $I_2^* \subset (\ell, q)$ then set $M = R/(m^2 + (\ell))$
- if $I_2^* \subset (T.C.)$ then set $M = R/(m^3 + (T.C.))$

Compute $P_M^R(t)$ explicitly and check that M exhibits non-polynomial Betti numbers.

New formula for Golod rings

Theorem (Avramov-S.-Yang)

If R is a CI and S is a Golod ring with $\varphi : R \twoheadrightarrow S$, then

$$P_S^R(t) = \frac{(1+t)((1-t^2)^a - 1) + tP_S^Q(t)}{t(1-t)^c(1+t)^{c-b}},$$

where

- *$Q \twoheadrightarrow S$ minimal Cohen presentation*
- *b =rank of the linear part of Ker(φ)*
- *a stems from comparing Cohen presentations for R and S*