The projective dimension of height two ideals generated by quadrics

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Stillman's Question

Question (Stillman)

Is there a bound, independent of N, on the projective dimension of ideals in $R = K[X_1, \ldots, X_N]$ which are generated by n homogeneous polynomials of given degrees $d_1, \ldots, d_n$?

This question is still open in full generality but has motivated

- obtaining general bounds for classes of ideals
- obtaining optimal or close-to-optimal bounds in specific situations

The case of ideals generated by quadrics (and cubics)

1. Ananyan-Hochster (2011, 2013+) Stillman's question has affirmative answer for ideals generated by quadratic polynomials (and cubic polynomials)
   - their bound is exponential in the number of quadrics: $O(2n^{2^n})$
   - doubly exponential for ideals generated by quadrics and cubics

2. small number of generators

   Eisenbud-Huneke I generated by 3 quadrics has $pd(R/I) \leq 4$
   Engheta I generated by 3 cubics has $pd(R/I) \leq 36$
   HMMS I generated by 4 quadrics has $pd(R/I) \leq 9$

   - compare the first bound above with 296 coming from A-H

Main result

Theorem (Huneke-Mantero-McCullough-S.)

For any ideal I of height two generated by n homogeneous quadratic polynomials in a polynomial ring R, $pd(R/I) \leq 2n-2$. Moreover, this bound is tight.

Example: the family of ideals

$$I = (x^2, y^2, a_{13}x - a_{23}y, \ldots, a_{1n}x - a_{2,n}y),$$

where $x, y, a_{1,1}, \ldots, a_{2,n-1}$ are distinct variables, has $pd(R/I) = 2n-2$.

Note: here I is generated by the minors of the matrix M below involving the first column

$$M = \begin{pmatrix} x & 0 & y & a_{13} & \ldots & a_{1n} \\ y & x & 0 & a_{23} & \ldots & a_{2n} \end{pmatrix}$$

Associated primes

A height two ideal generated by $n > 2$ quadrics I has $e(R/I) < 4$, hence it is contained in at least one prime $\mathfrak{p}$ of one of the following types:

1. a prime of multiplicity one and height two, i.e. $\mathfrak{p} = (x, y)$, with x, y independent linear forms,
2. a prime of multiplicity two and height two, i.e. $\mathfrak{p} = (x, q)$, with x a linear form and q an irreducible quadric or
3. a prime of multiplicity three and height two, i.e. the defining ideal of a variety of minimal multiplicity

Relation to matrices of linear forms

Consider the case of an ideal I generated by n quadrics and such that $I \subset (x, y)$, where x and y are linearly independent linear forms. Say that

$$I = \langle a_{21}x - a_{11}y, a_{22}x - a_{12}y, \dots, a_{2n}x - a_{11n}y \rangle.$$

Then we say I is represented by minors by the matrix

$$M = \begin{pmatrix} x & a_{11} & \dots & a_{1n} \\ y & a_{21} & \dots & a_{2n} \end{pmatrix}.$$

1-generic matrices

A matrix is called 1-generic if after applying any succession of K-linear row and column operations it exhibits no zero entries.

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$$\begin{bmatrix} x & z \\ y & w \end{bmatrix} \text{ is 1-generic} \qquad \begin{bmatrix} x & x \\ y & w \end{bmatrix} \text{ is not 1-generic}$$

1-generic matrices

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Theorem (Eisenbud)

If M is a 1-generic matrix of linear forms of size $p \times q$ ($p > q$), then the ideal generated by the maximal minors of M is prime of codimension $q - p + 1$.

Classification of matrices of linear forms of size $2 \times (n+1)$

Proposition

Any $2 \times (n+1)$ matrix of linear forms with $n \geq 2$ is equivalent via a sequence of ideal-preserving elementary operations to a matrix M' of one of the following types:

1. *M' is 1-generic;*
2. $M' = \begin{pmatrix} x & 0 & a_{12} & \dots & a_{1n} \\ y & a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix}$ *&* $\begin{pmatrix} x & a_{12} & \dots & a_{1n} \\ y & a_{22} & \dots & a_{2n} \end{pmatrix}$ *is 1-generic;*
3. $M' = \begin{pmatrix} x & 0 & 0 & a_{13} & \dots & a_{1n} \\ y & a_{21} & a_{22} & a_{23} & \dots & a_{2n} \end{pmatrix}$*, with no additional restrictions;*
4. $M' = \begin{pmatrix} x & 0 & a_{12} & a_{13} & \dots & a_{1n} \\ y & a_{21} & 0 & a_{23} & \dots & a_{2n} \end{pmatrix}$ *&* $\begin{pmatrix} x & a_{13} & \dots & a_{1n} \\ y & a_{23} & \dots & a_{2n} \end{pmatrix}$ *is 1-generic;*
5. $M' = \begin{pmatrix} x & 0 & a_{12} & a_{13} & a_{14} & \dots & a_{1n} \\ y & a_{21} & 0 & \lambda a_{13} & a_{24} & \dots & a_{2n} \end{pmatrix}$*, where λ is a scalar.*

Proof idea - for the 1-generic case

Suppose

$$I = \langle a_{21}x - a_{11}y, a_{22}x - a_{12}y, \ldots, a_{2n}x - a_{11n}y \rangle$$

and the matrix below is 1-generic

$$M = \begin{pmatrix} x & a_{11} & \ldots & a_{1n} \\ y & a_{21} & \ldots & a_{2n} \end{pmatrix}.$$

Then we use the following result

Theorem (Huneke)

Let C be a complete intersection containing an ideal $A = (a_1, \ldots a_s)$. Set $J = A : C$ and assume $\mathrm{ht}\, J \geq s$ and $\mathrm{ht}(C + J) \geq s + 1$.

$$\text{Then } \mathrm{ht}(J) = s \text{ and } pd(R/A) \leq s.$$

we use $A = I$, $C = (x, y)$, $s = n$ and Eisenbud's theorem on $I_2(M)$ to show

- $I_2(M) = I : (x, y) = I : (x) = I : (y)$ and
- $pd(R/I) \leq n$.

Open questions

Question

Let I be an ideal generated by n quadrics and having $\operatorname{ht} I = h$. Is there a sharp upper bound for $pd(R/I)$ expressed only in terms of n and h?

- answer is affirmative for $n \leq 3$ or $h \leq 2$

Question

Let I be an ideal generated by n quadrics and having $\operatorname{ht} I = h$. Is it true that

$$pd(R/I) \leq h(n - h + 1)?$$

- our main result answers this affirmatively for $h = 2$ and arbitrary n