Implicitization of tensor product surfaces via virtual projective resolutions

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Implicitization

A **tensor product surface** is the closed image of a rational map

\[ \lambda : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3 \]

The **implicitization problem** consists in finding the implicit equation of the image of \( \lambda \) given its parametrization.

**Example**

\[ \lambda : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3 \]

\[ [s : t] \times [u : v] \mapsto \left[ \begin{array}{c}
  s^2v \\
  stv \\
  stu \\
  t^2u
\end{array} \right] \]

\[ \Lambda = \text{image}(\lambda) = V(XW - YZ) \]
Implicitization

A tensor product surface is the closed image of a rational map

$$\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

The implicitization problem consists in finding the implicit equation of the image of $\lambda$ given its parametrization.

Example

$$\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

$$[s : t] \times [u : v] \mapsto [\frac{s^2 v}{X} : \frac{stv}{Y} : \frac{stu}{Z} : \frac{t^2 u}{W}]$$

$$\Lambda = \text{image}(\lambda) = \text{V}(XW - YZ)$$
Algebraic statement of implicitization

- $\mathbb{K}$ a field
- $R = \mathbb{K}[s, t; u, v]$ is the $\mathbb{Z}^2$-graded coordinate ring for $\mathbb{P}^1 \times \mathbb{P}^1$.
- $S = \mathbb{K}[X, Y, Z, W]$ is the coordinate ring for $\mathbb{P}^3$.
- $\lambda : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ defined by
  $$[s : t] \times [u : v] \mapsto [p_0 : p_1 : p_2 : p_3]$$

$$(X : Y : Z : W) \in \text{image}(\lambda) = \Lambda \iff \begin{cases} Xp_3 - Wp_0 = 0 \\ Yp_3 - Wp_1 = 0 \\ Zp_3 - Wp_2 = 0 \end{cases}$$

**Implicitization** amounts to computing

$$I_\Lambda = \langle Xp_3 - Wp_0, Yp_3 - Wp_1, Zp_3 - Wp_2 \rangle \cap S.$$
Resultants and residual resultants
Methods for implicitization

- **Gröbner bases** to find \( \langle Xp_3 - Wp_0, Yp_3 - Wp_1, Zp_3 - Wp_2 \rangle \cap S \)
  - are computationally expensive.

- **Resultants** to find image(\(\lambda\)) thought of as

\[
\{(X : Y : Z : W) \mid V(Xp_3 - Wp_0, Yp_3 - Wp_1, Zp_3 - Wp_2) \neq \emptyset\}
\]

  - fail in the presence of base points, where the set of base points of \(\lambda\) is \(V(p_0, p_1, p_2, p_3) \subseteq \mathbb{P}^1 \times \mathbb{P}^1\).

- **Residual resultants** – i.e. resultants that “remove” the base points are the focus of this talk
The image of a parametric surface can be thought of as

\[ \Lambda = \{(X : Y : Z : W) \mid V(\underbrace{Xp_3 - Wp_0}_f, \underbrace{Yp_3 - Wp_1}_f, \underbrace{Zp_3 - Wp_2}_f) \neq \emptyset\} \]

Definition

The resultant \( \text{Res}(f_0, f_1, f_2) \) is a homogeneous polynomial in the coefficients of \( f_0, f_1, f_2 \) that vanishes whenever the system \( f_i = 0 \) has a solution.

Hence \( \Lambda = \text{image}(\lambda) \subseteq V(\text{Res}(f_0, f_1, f_2)) \).
Residual resultant

Removing the base point locus $V(G)$ corresponds to looking for common vanishing of the elements of the \textbf{residual ideal} $F : G$.

$$\Lambda = \{(X : Y : Z : W) \mid V((f_0, f_1, f_2) : G) \neq \emptyset\}$$

**Definition (Busé–Elkladi–Mourrain)**

The \textbf{residual resultant} $\text{Res}_{G : \deg(f_i)}$ is a homogeneous polynomial in the coefficients of $f_0, f_1, f_2$ that vanishes whenever the system $f_i = 0$ has a solution outside $V(G)$.

**Theorem (Busé–Elkadi–Mourrain[2001], Busé[2001])**

The residual resultant exists if $G$ is (locally) a complete intersection ideal in a standard graded ring (in $\mathbb{P}^2$).
Residual resultant

Removing the base point locus $V(G)$ corresponds to looking for common vanishing of the elements of the **residual ideal** $F : G$.

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**Theorem (Busé–Elkadi–Mourrain[2001], Busé[2001])**

The residual resultant exists if $G$ is (locally) a complete intersection ideal in a standard graded ring (in $\mathbb{P}^2$).
Residual resultant for $\mathbb{P}^1 \times \mathbb{P}^1$

Let $G = \langle g_1, \ldots, g_n \rangle$, $F = \langle f_0, f_1, f_2 \rangle \subset R = \mathbb{K}[s, t; u, v]$ with $\deg g_i = (k_j, l_j), 1 \leq j \leq n$, $\deg f_i = (a_i, b_i), 0 \leq i \leq 2$

Theorem (Duarte – S. following Busé–Elkadi–Mourrain [’01])

Suppose

- $G \subseteq R$ is locally a complete intersection (e.g. the reduced ideal of a finite set of points) and
- $(a_i, b_i) \geq (k_{j_1} + 1, l_{j_1})$ and $(a_i, b_i) \geq (k_{j_2}, l_{j_2} + 1)$ for some $j_1, j_2$.

Then there exists a polynomial $\text{Res}_{G,\{(a_i,b_i)\}_{i=0}^2}$ which satisfies

$$\text{Res}_{G,\{(a_i,b_i)\}_{i=0}^2}(f_0, f_1, f_2) = 0 \iff \mathbb{V}(F : G) \neq \emptyset$$

and it has multihomogeneous degree in the coefficients of $f_k$

$$\deg \left( \text{Res}_{G,\{(a_i,b_i)\}_{i=0}^2}(f_0, f_1, f_2) \right) = \deg(f_i, f_j) - \deg(G).$$
Residual resultant for $\mathbb{P}^1 \times \mathbb{P}^1$

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\[ \deg \left( \text{Res}_{G, \{(a_i, b_i)\}_{i=0}^2} (f_0, f_1, f_2) \right) = \deg(f_i, f_j) - \deg(G). \]
Example

For the system below the base point locus in $\mathbb{P}^1 \times \mathbb{P}^1$ is $V(s, v)$.

$$\begin{cases}
  f_0 = (ua_{00} + va_{01})s + (sa_{02} + ta_{03})v \\
  f_1 = (ua_{10} + va_{11})s + (sa_{12} + ta_{13})v \\
  f_2 = (ua_{20} + va_{21})s + (sa_{22} + ta_{23})v
\end{cases}$$

The system has a solution outside $V(G)$, $G = \langle s, v \rangle$ whenever

$$\text{Res}_{G,(1,1)}(f_0, f_1, f_2) = \begin{vmatrix}
  a_{00} & a_{01} + a_{02} & a_{03} \\
  a_{10} & a_{11} + a_{12} & a_{13} \\
  a_{20} & a_{21} + a_{22} & a_{23}
\end{vmatrix} = 0$$

$$\deg \left( \text{Res}_{G,(1,1)}(f_0, f_1, f_2) \right) = 1 = \frac{1 \cdot 1 + 1 \cdot 1 - 1}{\deg(f_i, f_j)} - \frac{1}{\deg(G)} \text{ in } a_{k^*}.$$
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\end{vmatrix} = 0$$

$$\text{deg}\left(\text{Res}_{G,(1,1)}(f_0, f_1, f_2)\right) = 1 = \underbrace{1 \cdot 1 + 1 \cdot 1 - 1}_{\text{deg}(f_i, f_j) \text{ in } \mathbb{A}_k^*} - \underbrace{1}_{\text{deg}(G)}$$
Virtual resolutions
Virtual resolutions in $\mathbb{K}[\mathbb{P}^1 \times \mathbb{P}^1]$

$R = \mathbb{K}[s, t; u, v]$ is $\mathbb{Z}^2$-graded with $\begin{cases} \deg(s) = \deg(t) = (1, 0), \\ \deg(u) = \deg(v) = (0, 1) \end{cases}$, $B = \langle s, t \rangle \cap \langle u, v \rangle$ is geometrically irrelevant ideal of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition
A complex of free $\mathbb{Z}^2$-graded modules

$$F : \quad F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_m \leftarrow 0,$$

is called a **virtual resolution** if its homology groups $H_i(F)$ are $B$-torsion modules for $i > 0$.

- $M$ is $B$-torsion if $B^iM = 0$ for some $i$.
- Every free resolution is a virtual resolution.
## Projective vs. virtual resolutions

<table>
<thead>
<tr>
<th>Projective resolution / $K[\mathbb{P}^n]$</th>
<th>Proj. res. / $K[\mathbb{P}^1 \times \mathbb{P}^1]$</th>
<th>Virtual resolution / $K[\mathbb{P}^1 \times \mathbb{P}^1]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>length $\leq \dim(\mathbb{P}^n) + 1$</td>
<td>length $\leq 4$ $\times$</td>
<td>length $\leq \dim(\mathbb{P}^1 \times \mathbb{P}^1) + 1$ ✓</td>
</tr>
<tr>
<td>if $\dim(Z) = 0$ $\quad I_Z \leftarrow A^m \leftarrow A^{m-1} \leftarrow 0$</td>
<td>$\times$</td>
<td>if $\dim(Z) = 0$ $\quad I_Z$ has a Hilbert-Burch virtual resolution ✓</td>
</tr>
<tr>
<td>is a Hilbert-Burch resolution $\quad (A = k[x, y, z])$</td>
<td>$\times$</td>
<td>[Berkesch–Erman–Smith, 2017]</td>
</tr>
</tbody>
</table>
Projective vs. virtual resolution example

Example

\[ I_Z = \langle s, u \rangle \cap \langle t, v \rangle = \langle st, sv, tu, uv \rangle \]

\[
\begin{pmatrix}
-st & tu & sv & uv
\end{pmatrix}
\]

\[
\begin{pmatrix}
-u & -v & 0 & 0 \\
 s & 0 & 0 & -v \\
 0 & t & -u & 0 \\
 0 & 0 & s & t
\end{pmatrix}
\begin{pmatrix}
v \\
-u \\
-t \\
s
\end{pmatrix}
\]

\[ 0 \leftarrow I_Z \leftarrow R^4 \leftarrow R^4 \leftarrow R \leftarrow 0 \]

\[ 0 \leftarrow I_Z \cap B = G \leftarrow R^2 \leftarrow R^2 \leftarrow 0 \]
Towards a resolution of the residual ideal

Recall

- \( f_0 = Xp_3 - Wp_0, f_1 = Yp_3 - Wp_1, f_2 = Zp_3 - Wp_2 \)
- \( F = \langle f_0, f_1, f_2 \rangle \subseteq G = \langle g_1, \ldots, g_n \rangle, \ V(G) = \text{base point locus} \)

\[
\begin{pmatrix}
  h_{10} & h_{11} & h_{12} \\
  \vdots & \vdots & \vdots \\
  h_{n0} & h_{n1} & h_{n2}
\end{pmatrix}
\]

- (\( f_0, f_1, f_2 \)) = (g_1, \ldots, g_n)

- may assume 0 \( \leftarrow G \leftarrow R^n \xleftarrow{\varphi} R^{n-1} \leftarrow 0 \) is a Hilbert-Burch resolution.

Theorem (Buchsbaum-Eisenbud)

\[
\sqrt{F : G} = \sqrt{\text{Ann}(G/F)} = \sqrt{I_n \left[ \begin{array}{c}
  \varphi \\
  \psi
\end{array} \right]}.
\]
A virtual projective resolution

Let \( F = \langle f_0, f_1, f_2 \rangle \subseteq G = \langle g_1, \ldots, g_n \rangle \) as before.

**Theorem (Duarte – S.)**

*If for every point \( p \in \mathbb{P}^1 \times \mathbb{P}^1 \) there is an equality \( F_p = G_p \) then the Eagon-Northcott complex of the map \( \varphi \oplus \psi \) is a virtual projective resolution for \( I_n([\varphi \psi]) \).*

The hypothesis \( F_p = G_p \) for all \( p \in \mathbb{P}^1 \times \mathbb{P}^1 \) holds when

- the generators of \( F \) are generic linear combinations of the generators of \( G \) (with coefficients = new variables)
- the generators of \( F \) are general linear combinations of the generators of \( G \) (coefficients in a nonempty Zariski open set)
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Thank you!
Example

Let

\[ \alpha : \bigoplus_{i=1}^{n-1} R(-c_i, -d_i) \oplus R(-a, -b)^3 \to \bigoplus_{j=1}^{n} R(-e_j, -f_j) \]

and set \((c, d) = \sum_{i=1}^{n-1} (c_i, d_i)\) and \((e, f) = \sum_{j=1}^{n} (e_j, f_j)\). Then the graded shifts in the Eagon-Northcott complex of \(\alpha\) are:

<table>
<thead>
<tr>
<th>degree</th>
<th>shifts</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((a + c - e, b + d - f), (2a + c - e - c_i, 2b + d - f - d_i)) (, (3a + c - e - c_i - c_j, 3b + d - f - d_i - d_j), i \neq j)</td>
</tr>
<tr>
<td>2</td>
<td>((2a + c - e - e_j, 2b + d - f - f_j), (3a + c - e - c_i - e_j, 2b + d - f - d_i - f_j))</td>
</tr>
<tr>
<td>3</td>
<td>((3a + c - e - e_i - e_j, 3b + d - f - f_i - f_j))</td>
</tr>
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</table>