The Waldschmidt constant for squarefree monomial ideals

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CALIFORNIA

Outline

In this talk we'll focus on a squarefree monomial ideal I with primary decomposition $I = \bigcap_{i=1}^{s} P_i = \bigcap (x_{i_1}, \ldots, x_{i_{t_j}})$.

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- hypergraph
- (hyper-vertex) coloring
- fractional chromatic number

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Linear Programming

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Symbolic powers

Definition

The n-th **symbolic power** $I^{(n)}$ of an ideal $I \subset R$ is

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}(I)} I^n R_P \cap R$$

If I has no embedded primes then

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}(I)} P^n R_P \cap R = \bigcap_{P \in \mathrm{Ass}(I)} P^{(n)}$$

If I has no embedded primes and every P is a complete intersection

$$I^{(n)} = \bigcap_{P \in \mathrm{Ass}(I)} P^n$$

Growth of the α-invariant

Definition

For a homogeneous ideal J we denote by $\alpha(J)$ the smallest degree of an element in a minimal set of homogeneous generators for J.

Measuring the growth for symbolic powers:

- $\alpha(I^{(m)})$ measures the growth of the degrees of elements in $I^{(n)}$
- $\alpha(I^{(m)})$ is a sub-additive function: since $I^{(m_1+m_2)} \supseteq I^{(m_1)} I^{(m_2)}$,

$$\alpha(I^{(m_1+m_2)}) \leq \alpha(I^{(m_1)}) + \alpha(I^{(m_2)})$$

- given a subadditive function, $\lim_{n\to\infty} \frac{\alpha(I^{(m)})}{m} = \inf \frac{\alpha(I^{(m)})}{m}$ exists.

Waldschmidt constant

Definition

Given any homogeneous ideal I, the Waldschmidt constant of I is

$$\hat{\alpha}(I) := \lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n}.$$

- since $\alpha(I^{(n)}) \leq n\alpha(I)$, we have $\hat{\alpha}(I) \leq \alpha(I)$
- by Ein-Lazarsfeld-Smith, Hochster-Huneke, if $e =$big-height(I)

$$\begin{aligned} I^{(em)} &\subseteq I^m, \\ \alpha(I^{(em)}) &\geq m\alpha(I) \\ \hat{\alpha}(I) &\geq \frac{\alpha(I)}{e} \end{aligned}$$

Computing $\hat{\alpha}$ for a squarefree monomial ideal

Example: $I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$

$$x^a y^b z^c \in I^{(n)} = (x, y)^n \cap (x, z)^n \cap (y, z)^n$$

$$\Leftrightarrow \begin{cases} a + b \geq n \\ a + c \geq n \\ b + c \geq n \\ a, b, c \geq 0 \end{cases} \Leftrightarrow \begin{cases} \frac{a}{n} + \frac{b}{n} \geq 1 \\ \frac{a}{n} + \frac{c}{n} \geq 1 \\ \frac{b}{n} + \frac{c}{n} \geq 1 \\ \frac{a}{n}, \frac{b}{n}, \frac{c}{n} \geq 0 \end{cases}$$

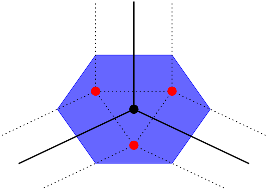


Figure: $\mathcal{Q}(I)$

$$\hat{\alpha}(I) = \min \left\{ \frac{a}{n} + \frac{b}{n} + \frac{c}{n} \,\middle|\, \left(\frac{a}{n}, \frac{b}{n}, \frac{c}{n} \right) \in \mathcal{Q}(I) \right\} = \frac{3}{2}$$

A linear programming approach

Lemma (BCGHJNSVV)

Let $I = P_1 \cap P_2 \cap \ldots \cap P_s$ be a squarefree monomial ideal and

$$A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i. \end{cases}$$

Then $\hat{\alpha}(I)$ is the optimum value of the LP

$$\begin{array}{rl} \text{minimize} & \mathbf{1}^T \mathbf{y} \\ \text{subject to} & A\mathbf{y} \geq \mathbf{1} \text{ and } \mathbf{y} \geq \mathbf{0}. \end{array}$$

In particular, for a monomial ideal, $\hat{\alpha}(I) \in \mathbb{Q}$.

Waldschmidt constant computed

... in two ways

1. as a limit

$$\hat{\alpha}(I) = lim_{n\to\infty} \frac{\alpha(I^{(n)})}{n} = \inf_{n\to\infty} \frac{\alpha(I^{(n)})}{n}$$

2. as the optimum value of a linear program

$$\begin{array}{rl} \textit{minimize} & \mathbf{1}^T \mathbf{y} \\ \textit{subject to} & A\mathbf{y} \geq \mathbf{1} \textit{ and } \mathbf{y} \geq \mathbf{0}. \end{array}$$

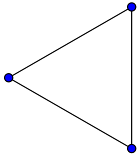
These two quantities are equal by our theorem.

Enter hypergraphs

Definition

There is a 1-to-1 corespondence between hypergraphs $H = (V, E)$ and squarefree monomial ideals $I(H)$ given by

$$\{x_{i_1}, \ldots, x_{i_t}\} \in E \iff x_{i_1} \cdots x_{i_t} \text{ is a minimal generator of } I(H).$$

Example: 

$$I(H) = (xy, xz, yz)$$

Fractional chromatic number

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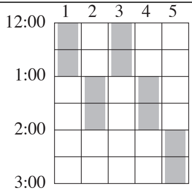
<td>
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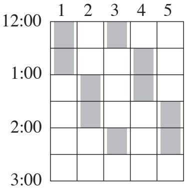
<td>
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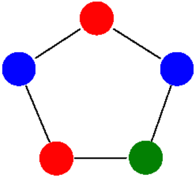
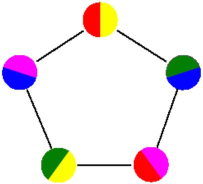
<td></td>
<td></td>
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<tr>
<td>Chromatic number</td>
<td>$\chi(C_5) = 3$</td>
<td>$\chi_f(C_5) = \frac{5}{2} = 2.5$</td>
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Fractional chromatic number defined

... in two ways

1. If H is a hypergraph with maximal independent sets $\{W_1, \ldots, W_t\}$, the fractional chromatic number $\chi_f(H)$ is the optimum value for

$$\begin{array}{rl} \textit{minimize} & \mathbf{1}^T\mathbf{y} \\ \textit{subject to} & B\mathbf{y} \geq \mathbf{1} \text{ and } \mathbf{y} \geq \mathbf{0}. \end{array}$$

where $B_{i,j} = \begin{cases} 1 & \text{if } x_i \in W_j \\ 0 & \text{if } x_i \notin W_j. \end{cases}$

2. $\chi_f(H) = lim_{b \to \infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b}$.

These two quantities are equal by general machinery.

Waldschmidt – fractional chromatic duality

Waldschmidt constant

If $I = P_1 \cap P_2 \cap \ldots \cap P_s$, then $\hat{\alpha}(I)$ = the optimum value for

$$\begin{array}{rl} \textit{minimize} & \mathbf{1}^T\mathbf{y} \\ \textit{subject to} & A\mathbf{y} \geq \mathbf{1} \\ & \mathbf{y} \geq \mathbf{0}. \end{array}$$

where $A_{i,j} = \begin{cases} 1 & \text{if } x_j \in P_i \\ 0 & \text{if } x_j \notin P_i. \end{cases}$

Fractional chromatic

If H is a hypergraph with maximal independent sets $\{W_1, \ldots, W_t\}$, $\chi_f(H)$ = the optimum value for

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where $B_{i,j} = \begin{cases} 1 & \text{if } x_i \in W_j \\ 0 & \text{if } x_i \notin W_j. \end{cases}$

Theorem (Bocci,Cooper,Guardo,Harbourne,Janssen,Nagel, S. ,VanTuyl,Vu)

$$\widehat{\alpha}(I) = \frac{\chi_f(H(I))}{\chi_f(H(I)) - 1}.$$

Consequences

Corollary (BCGHJNSVV)

Let G be a graph with chromatic number $\chi(G)$ and clique number $\omega(G)$ (thus $\omega(G) \leq \chi_f(G) \leq \chi(G)$).

(i) *Then*

$$\frac{\chi(G)}{\chi(G)-1} \leq \widehat{\alpha}(I(G)) \leq \frac{\omega(G)}{\omega(G)-1}.$$

(ii) *If G is a perfect graph, then $\widehat{\alpha}(I(G)) = \frac{\chi(G)}{\chi(G)-1}$.*

(iii) *If G is a complete k-partite graph, then $\widehat{\alpha}(I(G)) = \frac{k}{k-1}$.*

(iv) *If G is bipartite, then $\widehat{\alpha}(I(G)) = 2$.*

(v) *If $G = C_{2n+1}$ is an odd cycle, then $\widehat{\alpha}(I(C_{2n+1})) = \frac{2n+1}{n+1}$.*

(vi) *If $G = C_{2n+1}^c$, then $\widehat{\alpha}(I(G)) = \frac{2n+1}{2n-1}$.*