Configurations of points and lines and a question about symbolic powers

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Symbolic powers of ideals

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- in characteristic 0, this means the forms that vanish together with their first $n-1$ partial derivatives at every point of $\mathbf{X}$.

Comparing symbolic and ordinary powers

Theorem (Ein-Lazarsfeld-Smith, Hochster-Huneke)

For any homogeneous ideal $I \subseteq K[\mathbb{P}^N] = K[x_0, \ldots, x_N]$*, the following containment holds*

$$\mathbf{I}^{(\mathbf{Nr})} \subseteq \mathbf{I}^{\mathbf{r}}, \forall r \geq 1$$

proven by

- Ein-Lazarsfeld-Smith (2001), for I unmixed, using multiplier ideals
- Hochster-Huneke (2002) using reduction to characteristic p

Improving the containment

The theorem states that $I^{(Nr)} \subseteq I^r, \forall r \geq 1$. To make containment tighter:

- decrease the symbolic exponent Nr replacing it by $Nr - c$

  Question: Is there a c such that $I^{(Nr-c)} \subseteq I^r$ holds for all $r \geq 1$, for radical ideals I defining points? $(1 \leq c \leq N-1)$

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- multiply the ordinary power by powers of $\mathfrak{m} = (x_0, \dots, x_N)$
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A question about symbolic powers

Question (Huneke)

Does

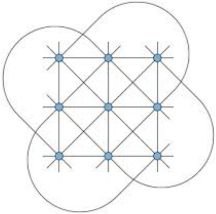
$$I^{(2\cdot 2-1)} = I^{(3)} \subseteq I^2$$

always hold in the case of $I \subseteq K[\mathbb{P}^2]$ *defining a reduced set of points of* $\mathbb{P}^2$*?*

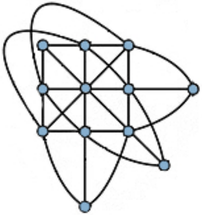
- Bocci-Harbourne: $I^{(3)} \subseteq I^2$ holds for points in general position in $\mathbb{P}^2$.

Three classical configurations

Hesse configuration



Fermat-3 configuration



Klein configuration

49 pts & 21 lines
21 quadruple
28 triple

realizable over

$K[a]/(a^2 + a + 2)$
e.g. $\mathbb{R}[\sqrt{-7}]$
or $\mathbb{Z}/7$

Wiman configuration

201 pts & 45 lines
36 quintuple
45 quadruple
120 triple

realizable over

$K[a]/(a^4 - a^2 + 4)$
e.g. $\mathbb{Z}/19$
or $\mathbb{Z}/31$

Counterexamples from classical configurations

- Dumnicki, Szemberg and Tutaj-Gasińska
  the Fermat-3 configuration has $I^{(3)} \not\subset I^2$ over $\mathbb{C}$

$$I = (x_0(x_1^3 - x_2^3), x_1(x_0^3 - x_2^3), x_2(x_0^3 - x_1^3))$$

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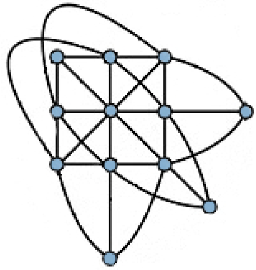
  suggest the Klein & Wiman configurations have $I^{(3)} \not\subset I^2$
- S. gives a proof that the Fermat, Klein & Wiman configurations have

$$I^{(3)} \not\subset I^2$$

Geometric intuition



Let F =the product of all the lines in the configuration

- $F \in I^{(3)}$ is easy to see: every point is a triple point
- $F \notin I^2$ is much harder to prove

Same is true for all of the counterexamples (Fermat-n, Klein, Wiman).

A homological criterion for detecting counterexamples

Theorem (S., 2014)

Let $I = (f, g, h)$ be a homogeneous ideal with minimal generators of the same degree d, defining a reduced set of points in $\mathbf{P}^2$ over a field of characteristic not equal to 3. Then:

- *the minimal free resolution of I^3 has the form*

$$0 \longrightarrow R^3 \xrightarrow{\mathbf{Y}} R^{12} \longrightarrow R^{10} \longrightarrow I^3 \longrightarrow 0,$$

- *if* $\begin{bmatrix} f \\ g \\ h \end{bmatrix} \notin \mathrm{Image}(\mathbf{Y}^{\mathbf{T}})$ *then* $\mathbf{I}^{(3)} \not\subset \mathbf{I}^{2}$.

This applies to the Fermat, Klein, Wiman configurations showing $I^{(3)} \not\subset I^2$.

Relations with curves of high negative self-intersection

The *linear H-constant* of a configuration of points $\mathcal{P} = \{P_1, \ldots, P_s\}$ which is the singular locus for a line configuration $\mathcal{L} = \{L_1, \ldots, L_d\}$ is

$$H(\mathcal{P}, \mathcal{L}) = \frac{d^2 - \sum_{i=1}^{s} m_{P_i}^2}{s}.$$

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On $\mathbb{P}^2$, $H(\mathcal{P}, \mathcal{L}) \geq -4$. The closest values to the bound known are

Configuration	$H(\mathcal{P}, \mathcal{L})$
Fermat-n	$-3n^2/(n^2+3) \to -3$
Klein	-3
Wiman	-3.36

Question: Is there a connection between configurations of high negative self-intersection and counterexamples to $I^{(3)} \subset I^2$?

Open questions

1. The most conservative version

   Is it true that $I^{(Nr-1)} \subseteq I^r$ holds for all radical ideals I of finite sets of points in $\mathbb{P}^N$ for all $r \geq 1$ as long as $N > 2$?

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2. Revised version of C. Huneke's question:

   Is it always true for the ideal I of a finite set of points in $\mathbb{P}^3$ that $I^{(5)} \subseteq I^2$?