The maximum determinant of (± 1)-matrices

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Abstract

We give a new proof for the bound on the value of the determinant of a (± 1)-matrix of dimension $n \equiv 1 \pmod 4$ first given in [Ba]. Adapting a construction of A.E. Brouwer we give examples to show that the bound is sharp for infinitely many values of n. This in turn gives an infinite family of examples which attain the bound given by H. Ehlich ([Eh1]) and M. Wojtas ([Wo]) for the determinant of a (± 1)-matrix of dimension $n \equiv 2 \pmod 4$. For $n \equiv 3 \pmod 4$ we construct an infinite family of examples which attain slightly more than 1/3 of the bound given in [Eh2].

1 Introduction and the main result

Let $\mathcal{H}_n$ be the set of (± 1)-matrices of dimension n. The question of the maximum value of the determinant of an element $N \in \mathcal{H}_n$ is an old one which

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goes back to the beginnings of matrix theory. It is a simple consequence of Hadamard's inequality [Hd] that for all $M \in \mathcal{H}_n$

$$\det M \leq (\prod_{i=1}^{n} \sum_{j=1}^{n} m_{ij}^2)^{1/2} \leq n^{n/2}. \tag{1}$$

There is a large body of work addressing the question of when (1) is sharp.

Definition 1.1 Matrices in $\mathcal{H}_n$ for which equality holds in (1) are called *Hadamard matrices.*

For an $n \times n$ Hadamard matrix to exist it is necessary that n be either 1, 2, or a multiple of 4, and it is conjectured that this condition is also sufficient. According to [AK] the smallest value for which the existence of a Hadamard matrix is in question is $n = 4 \cdot 107 = 428$.

Hadamard matrices do exist for many infinite families of values of n, e.g. Sylvester ([Syl]) proved that Sylvester's matrix,

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

tensored with itself t times, gives a Hadmard matrix of dimension 2^t. The Paley construction (see e.g. [AK], p. 271-275, or [vLW]) for an odd prime power p gives a Hadamard matrix of size $p+1$ if $p \equiv 3 \pmod 4$ and size $2p+2$ if $p \equiv 1 \pmod 4$, thus providing another infinite family of Hadamard matrices.

For further discussion of Hadamard matrices the reader might consult [Bo] or one of the surveys [Ag] or [SY].

1.1 $(0,1)$-Matrices

There is a strong connection between $\mathcal{H}_n$ and $\mathcal{Z}_{n-1}$, the set of $(0,1)$-matrices of dimension $n-1$. Specifically, there exists an injection ϕ from $\mathcal{Z}_{n-1}$ into $\mathcal{H}_n$, which preserves the relative sizes of determinants. The function ϕ can be described as follows:

Given M in $\mathcal{Z}_{n-1}$ let M' be the $n \times n$ block matrix

$$M' = \begin{pmatrix} 1 & 1 \\ 0 & -2M \end{pmatrix}$$

$\phi(M)$ is obtained by adding the first row of M' to each other row. The resulting matrix is clearly a (± 1)-matrix and the passage from M' to $\phi(M)$ doesn't change the value of the determinant. Thus it is clear that

$$\det \phi(M) = (-2)^{n-1} \det M.$$

Let J_n denote the $n \times n$ matrix all of whose entries are 1. The map ϕ can also be described as follows:

$$\phi(M) = J_n - \begin{pmatrix} 0 & 0 \\ 0 & 2M \end{pmatrix}.$$

The image of ϕ is the set $\mathcal{H}'_n$ of matrices $N = (n_{ij})$ in $\mathcal{H}_n$ such that $n_{1j} = 1$ for all $1 \leq j \leq n$ and $n_{i1} = 1$ for all $1 \leq i \leq n$. The set $\mathcal{H}'_n$ is a sufficiently large subset of $\mathcal{H}_n$ for our purposes since for every matrix $M \in \mathcal{H}_n$ we can find suitable diagonal matrices $P = (p_{ij})$ and $Q = (q_{ij})$ with $p_{ii}, q_{ii} \in \{\pm 1\}$ such that $PMQ \in \mathcal{H}'_n$. Since $\det P = \pm 1 = \det Q$ we have $|\det M| = |\det PNQ|$.

If we write $m(\mathcal{C})$ for $\max\{\det M | M \in \mathcal{C}\}$ then the above remarks show that

$$m(\mathcal{H}_n) = m(\mathcal{H}'_n) = 2^{n-1}\, m(\mathcal{Z}_{n-1}).$$

In particular $m(\mathcal{Z}_{n-1}) \leq n^{n/2}/2^{n-1}$ and the $(0,1)$-matrices which attain this bound are the pre-images under ϕ of Hadamard matrices. From another point of view they can be regarded as the incidence matrices of Hadamard designs; a class of symmetric designs with parameters $(4m-1, 2m-1, m-1)$. (See [AK], [Bo], or [Ag] for more details.)

1.2 What is known when $n \not\equiv 0 \pmod 4$?

The first reference to the case $n \not\equiv 0 \pmod 4$ seems to be Colucci ([Col]) in 1926. The following lemma, concerning the case $n \equiv 1 \pmod 2$, was discussed by Barba ([Ba]), and proved, independently of Barba, by Ehlich ([Eh1], Satz 4.1).

Lemma A:[Barba, Ehlich] *Suppose that* $n \equiv 1 \pmod 2$*. For all* $N \in \mathcal{H}_n$

$$\det N \quad \leq \quad (2n-1)^{1/2}(n-1)^{(n-1)/2}. \tag{2}$$

I.e., $m(\mathcal{H}_n) \leq (2n-1)^{1/2}(n-1)^{(n-1)/2}$. *Equivalently, for all* $M \in \mathcal{Z}_{n-1}$

$$2^{n-1} \det M \quad \leq \quad (2n-1)^{1/2}(n-1)^{(n-1)/2}. \tag{3}$$

In 1937 T. Popoviciu ([Po]), apparently unaware of [Ba], gave a weaker bound. Curiously enough, J. Brenner [Bre] claims Popoviciu's bound to be sharper than (2). Neither [Ba] nor [Po] address the case $n \equiv 2 \pmod 4$ separately but rather use Hadamard's bound for $n \equiv 0 \pmod 4$ in this case as well, though it was known that Hadamard's bound can only be attained for $n = 1, 2$ or $n \equiv 0 \pmod 4$.

It seems that H. Ehlich and, independently, M. Wojtas were the first to address the case $n \equiv 2 \pmod 4$ ([Eh1] and [Wo]). They also seem to have been the first to address the question of the structure of matrices of maximal determinants.

Proposition A:[Ehlich, Wojtas] *Suppose that* $n \equiv 1 \pmod 4$. *For all* $N \in \mathcal{H}_n$ *inequality (2) holds and in order for equality to hold in (2) it is necessary that* $2n-1$ *be a square and that there exists an* $N \in \mathcal{H}_n$ *with* $NN^T = (n-1)I_n + J_n$.

Proposition B:[Ehlich, Wojtas] *Assume* $n \equiv 2 \pmod 4$. *For all* $N \in \mathcal{H}_n$

$$\det N \quad \leq \quad (2n-2)(n-2)^{\frac{1}{2}n-1} \tag{4}$$

Moreover, equality in (4) holds if and only if there exists $N \in \mathcal{H}_n$ *such that*

$$NN^T = N^TN = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix},$$

where $L = (n-2)I + 2J$ *is a* $\frac{n}{2} \times \frac{n}{2}$ *matrix. A further necessary condition for equality to hold is that* $2n-2$ *is the sum of two squares.*

H.S.E. Cohn gave an independent proof of Lemma B above and provided further information on the structure of maximal examples ([Co2], Theorem 2).

In a sequel to [Eh1] Ehlich investigated the case $n \equiv 3 \pmod 4$ ([Eh2]).

Proposition C:[Ehlich] *Assume* $n \equiv 3 \pmod 4$ *and* $n \geq 63$. *For all* $N \in \mathcal{H}_n$

$$\det N \quad \leq \quad \sqrt{\frac{4 \cdot 11^6}{7^7}(n-3)^{n-7}n^7} \tag{5}$$

Moreover, for equality to hold it is necessary that $n = 7m$ *and that there exists* $N \in \mathcal{H}_n$ *with*

$$NN^T = I_7 \otimes ((n-3)I_m + 4J_m) - J_n.$$

The corresponding bounds for all values $n \equiv 3 \pmod 4$, $n < 63$, are also given in [Eh2] and the structures of NN^T for normalized maximal examples N is also given. The theme for values $n < 63$ is the same as for the above example: A (± 1)-matrix N has maximal determinant if NN^T has block structure with the blocks along the diagonal of the form $(n-3)I + 3J$ and the off-diagonal blocks equal to $-J$.

Though it is well-known that the Hadamard bound in the case $n \equiv 0 \pmod 4$ is attained infinitely often and has to be considered sharp in this sense, it was not known if the bounds given in Proposition A, Proposition B and Proposition C are sharp in this sense. The object of this paper is to show that this is indeed the case for $n \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$. For $n \equiv 3 \pmod 4$ we provide an infinite family of examples which asymptotically have determinants slightly larger than $1/3$ of the bound of Proposition C. The case $n \equiv 1 \pmod 4$ is crucial to everything and so we provide our own version and proof of Proposition A to keep the exposition self-contained. We then extend the known results to show that the bound of Proposition A is sharp, i.e. it is attained infinitely often.

Theorem A: *Suppose that* $n \equiv 1 \pmod 4$. *For all* $N \in \mathcal{H}_n$ *inequality (2) holds. In order for equality to hold in (2) it is necessary that* $2n-1$ *be a square and that* $NN^T = (n-1)I_n + R$ *where* $\operatorname{rank} R = 1$ *and* $|r_{ij}| = 1$ *for all* i, j.

Furthermore, if $n = 2(q^2+q)+1$ *for some odd prime power* q, *then there exist matrices* $N \in \mathcal{H}_n$ *for which equality holds in (2), i.e.*

$$m(\mathcal{H}_n) \;=\; (2n-1)^{1/2}(n-1)^{(n-1)/2} \tag{6}$$

for all $n = 2(q^2+q)+1$, q *an odd prime power. In particular the bound in (2) is sharp for infinitely many values of* n.

The proof of Theorem A naturally divides into two parts. In section 2 we prove inequality (2). In section 3 we build an infinite family of examples to prove that

$$\mathcal{B}_n = \{N \in \mathcal{H}_n \mid n \equiv 1 \pmod 4 \text{ and } N \text{ attains equality in (2)}\}$$

is non-empty whenever $n = 2q^2 + 2q + 1$ and q is an odd prime power.

I. Kaplansky [Ka2] claims that the bound in (2) can be attained only if there exists a design with parameters $(2q^2 + 2q + 1, q^2, (q^2 - q)/2)$.

The particular values $n = 17$ and $n = 21$ for which the bound cannot be attained are discussed in [MK] and [CKM] respectively.

The examples that we construct can be used to give an infinite family of examples which attain the bound of Proposition B.

Theorem B: *If $n = 4(q^2 + q) + 2$ for some odd prime power q, then there exists a matrix $N \in \mathcal{H}_n$ for which equality holds in inequality (4) of Theorem A. In particular, equality holds in (4) for infinitely many values of n.*

This was already observed by Whiteman ([Wh]) who gave the identical proof as given below. He presented his results in the context of D-optimal designs. For a definiton of D-optimal designs and other optimality criteria see [SS]. In [KKS] another infinite family of matrices is given whose determinants attain the bound in (4).

Proof [of Theorem B] Suppose $n \equiv 1 \pmod 4$ and that $N \in \mathcal{B}_n$. After a suitable change of basis we may assume, by Theorem A, that $NN^T = (n-1)I_n + J_n$. Now define

$$\tilde{N} = S \otimes N = \begin{pmatrix} N & N \\ N & -N \end{pmatrix} \tag{7}$$

where S is Sylvester's matrix defined above. It is easy to show that

$$\tilde{N}\,\tilde{N}^T = \begin{pmatrix} (2n-2)I_n + 2J_n & 0 \\ 0 & (2n-2)I_n + 2J_n \end{pmatrix} \tag{8}$$

which, implies that $\det \tilde{N} = (2n-2)(n-2)^{\frac{1}{2}n-1}$ ([Eh1], [Wo] or [Co2]). ∎

Unfortunately, we were unable to construct an infinite family of examples which attain the bound of Proposition C. In fact, we do not even know of a single example for which the bound in Proposition C is attained. Nevertheless, our efforts yielded an infinite family of examples whose determinants assymptotically take on values slightly larger than 1/3 of the bound of Proposition C. This shows that the bound of Proposition C is of the correct order. This is a significant improvement over the lower bound given in [CL].

Theorem C: *There exists an infinite family of examples* $N_n \in \mathcal{H}_n$ *such that*

$$\lim_{n\to\infty} \frac{\det\, N_n}{h_n} = \frac{1}{2}\left(\frac{7}{11}\right)^3 \sqrt{7} \approx 0.34$$

where $h_n^2 = \frac{4\cdot 11^6}{7^7}(n-3)^{n-7}n^7$.

A proof of Theorem C is given in section 4.

At this point it remains unresolved if the bound in (5) can ever be attained. We have no doubt that for a particular value of $n \equiv 3 \pmod 4$ an efficient computer search is likely to produce examples whose determinants have larger values than the ones given by us in section 4. The real challenge, however, is to find an infinite family of examples of examples whose determinants take on the bound in (5) or to show that the bound in (5) can be improved. It is conceivable that it is not sharp. The methods used by Ehlich in [Eh2] study the maximum of the values of the determinant on a subset of positive definite matrices. This subset is defined by certain necessary condition its elements have to satisfy if they are to be of the form NN^T for some $N \in \mathcal{H}_n$. It is certainly possible that this subset contains many positive definite matrices which are not of the form NN^T for some $N \in \mathcal{H}_n$. Hence the bound derived from this subset might be larger than the actual bound.

Having said that, we mention that B. Solomon ([Sol]) has provided us with examples for $n = 11$, $n = 15$ and $n = 19$, believed on the basis of computer experiments to be maximal, which seem to show that maximal examples tend to exhibit the behavior forecast in [Eh2].

Some interesting comments on determinants of (± 1)-matrices in the context of D-optimal designs can be found in [GK].

2 The proof of the inequality of Theorem A

The next lemma is proved in [Co1]. For completeness we include a (slightly shorter) proof.

Lemma 2.1 *Suppose that* $T \in M_n(\mathbf{R})$ *is a symmetric, diagonalizable matrix with the following properties: it has 0 on the diagonal,* $I_n + T$ *is positive definite, and the root mean square of its off-diagonal elements is* c*. Then*

$$\det(I_n + T) \leq (1 + c(n-1))(1-c)^{n-1} \tag{9}$$

Proof Let $z_1, z_2, \ldots, z_n$ be the eigenvalues of T. We know that $\sum_{i=1}^n z_i = 0$ and

$$\sum_{i=1}^{n} z_i^2 = \sum_{i,j} t_{ij}^2 = n(n-1)c^2 \tag{10}$$

Split the eigenvalues into two classes according to their sign; let x_1, x_2, $\ldots, x_k$ be the non-negative ones, y_1, y_2, $\ldots, y_{n-k}$ the negative ones. Set $\bar{x} = (\sum_1^k x_i)/k$, and $\bar{y} = (\sum_1^{n-k} y_i)/(n-k)$. Now

$$\begin{aligned} \det(I_n + T) &= \prod_{i=1}^{k}(1+x_i) \cdot \prod_{i=1}^{n-k}(1+y_i) \\ &\leq (1+\bar{x})^k (1+\bar{y})^{n-k}, \end{aligned} \tag{11}$$

where the inequality is just the Arithmetic Mean–Geometric Mean Inequality. Thus if we define $g : (k, x, y) \mapsto k\ln(1+x) + (n-k)\ln(1+y)$ we have that $\ln\det(I_n+T) \leq g(k, \bar{x}, \bar{y})$. We want a bound on $g(k, \bar{x}, \bar{y})$ given that $k\bar{x} + (n-k)\bar{y} = 0$ and $k\bar{x}^2 + (n-k)\bar{y}^2 = n(n-1)c^2$. For convenience we denote this last quantity by S. Solving for $\bar{x}$ and $\bar{y}$ in terms of k we get

$$\bar{x} = \sqrt{S}\frac{\sqrt{n-k}}{\sqrt{n}\sqrt{k}} \qquad \bar{y} = -\sqrt{S}\frac{\sqrt{k}}{\sqrt{n}\sqrt{n-k}}.$$

Therefore the quantity we wish to estimate is

$$g(k) = k\ln\left(1 + \sqrt{S}\frac{\sqrt{n-k}}{\sqrt{n}\sqrt{k}}\right) + (n-k)\ln\left(1 - \sqrt{S}\frac{\sqrt{k}}{\sqrt{n}\sqrt{n-k}}\right).$$

The function $g(k)$ is of course defined for all real $k \in (0, n)$, but condition (10) ensures that $k \in [1, n-1]$. We will show that $g(k)$ is decreasing on $(0, n)$ Indeed, differentiating we get (writing $\bar{x}$ and $\bar{y}$ for the above functions of k),

$$g'(k) = \ln(1+\bar{x}) - \ln(1+\bar{y}) - \frac{1}{2}(\bar{x}-\bar{y})\left(\frac{1}{1+\bar{x}} + \frac{1}{1+\bar{y}}\right) \tag{12}$$

To show that $g'(k) < 0$ we need only prove that

$$\frac{1}{\bar{x}-\bar{y}}\int_{\bar{y}}^{\bar{x}} \frac{dt}{1+t} < \frac{1}{2}\left(\frac{1}{1+\bar{x}} + \frac{1}{1+\bar{y}}\right).$$

This is an immediate consequence of the strict convexity of the function $t \mapsto 1/(1+t)$. Therefore, for all $k \in [1, n-1]$, $g(k) \leq g(1)$ where

$$\begin{aligned} g(1) &= \ln\left(1 + \sqrt{S}\sqrt{1 - 1/n}\right) + (n-1)\ln\left(1 + \sqrt{S}\sqrt{1/n(n-1)}\right) \\ &= \ln(1 + c(n-1)) + (n-1)\ln(1-c) \end{aligned} \tag{13}$$

The result follows immediately.

∎

Next we prove a result which is analogous to Lemma 1 of [Co2]. It is slightly more general than is needed here but the proof makes the result natural. Define the function

$$f : (n, h) \mapsto (n + h - 1)(h - 1)^{(n-1)} \tag{14}$$

Lemma 2.2 *Suppose that* M *is a positive definite symmetric matrix of the form* $(h-1)I_n + R \in M_n(\mathbf{R})$*, where* $h > 1$ *and* $R = (r_{ij})$ *has* $|r_{ij}| \geq 1$ *and* $r_{ii} = 1$*. Then*

1. $|\det M| \leq f(n, h)$.
2. *Equality holds if and only if* $|r_{ij}| = 1\ \forall\, 1 \leq i, j \leq n$ *and* $\operatorname{rank} R = 1$.

Proof First notice that if $M = (h-1)I_n + R$ with R as in part 2, then

$$\begin{aligned} \det M &= (h-1)^n\left(1 + \frac{1}{h-1}\operatorname{trace}(R)\right) \\ &= (h-1)^n\left(1 + \frac{n}{h-1}\right) \\ &= (h-1)^{n-1}(h - 1 + n) = f(n, h) \end{aligned}$$

We prove the theorem by induction on the dimension n. The case $n = 2$ is a straightforward computation. Assume then that $n > 2$ and that the result holds for smaller values of n.

Now $\det M = \det((h-1)I_n + R) = \det(hI_n + (R - I_n)) = h^n \det(I_n + h^{-1}(R - I_n))$ and the matrix $T = h^{-1}(R - I_n)$ satisfies the conditions of Lemma 2.1 with $c^2 = \frac{1}{n(n-1)} \sum_{i \neq j} r_{ij}^2/h^2 \geq 1/h^2$. Thus

$$\det M = h^n \det(I_n + T)$$

$$\begin{aligned} &\leq h^n \left(1 + \frac{n-1}{h}\right)(1-1/h)^{n-1} \\ &= (n+h-1)(h-1)^{n-1} \\ &= f(n,h) \end{aligned} \tag{15}$$

It is clear that if $|m_{ij}| > 1$ for some pair (i,j), $i \neq j$, then $c > 1/h$ and the inequality becomes strict. Thus for equality to occur it is necessary that $|m_{ij}| = 1$ for all $1 \leq i,j \leq n$.

After a suitable base change we may assume that the entries of the last row and the last column of M are all equal to 1, i.e. $m_{in} = m_{nj} = 1$ for all $1 \leq i,j \leq n$.

Now subtract $1/h$ times the last row from the first $n-1$ rows and then subtract $1/h$ times the last column from the first $n-1$ columns. The resulting matrix is

$$M' = \begin{pmatrix} M_1 & 0 \\ 0 & h \end{pmatrix},$$

where $M_1 = (m'_{ij})$, $1 \leq i,j \leq n-1$, is a symmetric, positive definite matrix (of size $(n-1) \times (n-1)$) such that

$$(1-1/h)^{-1} m'_{ij} = \begin{cases} h+1 & \text{if} \quad i = j \\ 1 & \text{if} \quad m_{ij} = 1 \\ -\frac{h+1}{h-1} & \text{if} \quad m_{ij} = -1 \end{cases} \tag{16}$$

Note that $(h+1)/(h-1) > 1$. Since $\det M = f(n,h)$ and $\det M = h \cdot \det M_1$ we see that

$$\det M_1 = \frac{f(n,h)}{h} = \frac{(h+n-1)(h-1)^{n-1}}{h} \tag{17}$$

On the other hand, applying the induction hypothesis to the matrix $\tilde{M}_1 = (1-1/h)^{-1} M_1$ we get

$$\begin{aligned} \det M_1 &\leq (1-1/h)^{n-1}(h+n-1)h^{n-2} \qquad (18) \\ &= \frac{(h+n-1)(h-1)^{n-1}}{h}. \qquad (19) \end{aligned}$$

Thus $\det \tilde{M}_1 = f(n-1, h+1)$. By induction this implies that $\tilde{M}_1 = hI_{n-1} + \tilde{R}$ where $\operatorname{rank} \tilde{R} = 1$ and $|\tilde{r_{ij}}| = 1$. Hence, by (16), we conclude that $m_{ij} = 1$ for all $1 \leq i,j \leq n$, i.e. $P = (h-1)I_n + R$ with $\operatorname{rank} R = 1$ and $|r_{ij}| = 1$. ∎

The statements of Theorem A pertaining to the inequality follow from Lemma 2.2. We summarize this in the next result.

Corollary 2.3 *Let* $N \in \mathcal{H}_n$. *Then* $|\det N| \leq \sqrt{f(n,n)}$ *with equality if and and only if* $NN^T = (n-1)I_n + R$ *with* $|r_{ij}| = 1$ *and rank* $R = 1$.

Proof Instead of working with the (± 1)-matrix N we consider the matrix $M = NN^T$. If N is singular there is nothing to prove. If N is invertible then, because n is odd, it follows that

$$M = (n-1)I_n + R$$

is symmetric, positive definite, with $|r_{ij}| \geq 1$. Since $\det M = (\det N)^2$ the corollary now follows from lemma 2.2 ∎

3 An infinite family of maximal examples

In this section we construct an infinite family of (± 1)-matrices which yield equality in (2). Using the function ϕ we can construct from these an infinite family of $(0,1)$-matrices which yield equality in (3). The construction is very close to one by A.E. Brouwer ([Bro]); we include the details for completeness.

Recall that, by Proposition A, we are trying to construct, for some $n \equiv 1 \pmod 4$, a matrix $N \in \mathcal{H}_n$ such that $N^T N = (n-1)I_n + J_n$. We will show that this is possible whenever $n = 2q^2 + 2q + 1$ and q is an odd prime power.

Down to work. Let q be an odd prime power. By the Paley construction ([vLW]) there exists a Hadamard matrix H_{2q+2} of size $2q+2$, which we may assume is normalised; i.e., its first row and column are all 1s. We use H_{2q+2} to define matrices E and A, of dimension $q \times 2q$ and $(q+1) \times 2q$ respectively: after a suitable rearrangement of rows we may assume that

$$H_{2q+2} = \begin{pmatrix} 1 & 1 & 1 \ldots 1 \\ 1 & 1 & \\ \vdots & \vdots & E \\ 1 & 1 & \\ 1 & -1 & \\ \vdots & \vdots & A \\ 1 & -1 & \end{pmatrix}.$$

The matrices E and A are (± 1) matrices satisfying the following (where $J_{m,l}$ is an $m \times l$ matrix of 1s):

$$\begin{aligned} J_{m,q}E &= -J_{m,2q} && (20) \\ EJ_{2q,m} &= -2J_{q,m} && (21) \\ J_{m,q+1}A &= 0 && (22) \\ AJ_{2q,m} &= 0 && (23) \\ E^T E + A^T A &= (2q+2)I_{2q} - J_{2q} && (24) \end{aligned}$$

Let C be a $(q+1) \times (q+1)$ conference matrix. That is, C is a $\{0, \pm 1\}$-matrix with $C_{ij} = 0 \iff i = j$, and

$$C^T C = CC^T = qI. \qquad (25)$$

Such a matrix can be constructed, for instance, by enumerating the elements $\mathbf{F}_q = \{x_1, x_2, \ldots, x_q\}$, the field with q elements, then adding a row and column of 1s to the matrix $(\chi(x_i - x_j))_{i,j=1}^n$ and finally changing the first entry on the diagonal back to 0. Here $\chi : F_q \to \{0, \pm 1\}$ is the quadratic character.

Let L be the $q^2 \times q(q+1)$ incidence matrix of points and lines in affine $\mathbf{F}_q^2$. The lines in $\mathbf{F}_q^2$ can be split into $q+1$ parallel classes according to their slope and two lines are disjoint if they are parallel, and meet in one point if they are not. Thus if we group the columns of L by their slope we find that L is a $\{0, 1\}$-matrix satisfying

$$\begin{aligned} L^T L &= (J_{q+1} - I_{q+1}) \otimes J_q + I_{q+1} \otimes qI_q && (26) \\ J_{m,q^2} L &= qJ_{m,q(q+1)}. && (27) \end{aligned}$$

We are now ready to give the example promised; define

$$N_q = \begin{pmatrix} 1 & \\ \vdots & L(I_{q+1} \otimes E) \\ 1 & \\ -1 & \\ \vdots & C \otimes A - I_{q+1} \otimes J_{q+1,2q} \\ -1 & \end{pmatrix} \qquad (28)$$

It is straightforward to check that $N_q \in \mathcal{H}_{2q^2+2q+1}$. First let's see that the first column of N_q has inner product 1 with each of the others; simply note that

$$\begin{aligned} J_{1,q^2} L(I_q \otimes E) &= qJ_{1,q(q+1)}(I_{q+1} \otimes E) \\ &= J_{1,q+1} \otimes (qJ_{1,q}E) \\ &= J_{1,q+1} \otimes (-qJ_{1,2q}) \\ &= -qJ_{1,2q^2+2q} \end{aligned} \tag{29}$$

and

$$\begin{aligned} -J_{1,(q+1)^2}(C \otimes A - I_{q+1} \otimes J_{q+1,2q}) &= -(J_{1,q+1}C) \otimes (J_{1,q+1}A) \\ &\quad\; +J_{1,q+1} \otimes (q+1)J_{1,2q} \\ &= (q+1)J_{1,2q^2+2q} \end{aligned} \tag{30}$$

Now let M_q be N_q with its first column removed. We want to prove that $M_q^T M_q = (2q^2+2q)I_{2q^2+2q} + J_{2q^2+2q}$. We calculate as follows, using the properties of E, A, C, and L established above in (20) through (30).

$$\begin{aligned} M_q^T M_q &= (I_{q+1} \otimes E^T)L^T L(I_{q+1} \otimes E) \\ &\qquad +(C^T \otimes A^T - I_{q+1} \otimes J_{2q,q+1})(C \otimes A - I_{q+1} \otimes J_{q+1,2q}) \\ &= (I_{q+1} \otimes E^T)((J_{q+1} - I_{q+1}) \otimes J_q + I_{q+1} \otimes qI_q)(I_{q+1} \otimes E) \\ &\qquad +C^T C \otimes A^T A - C \otimes (J_{2q,q+1}A) - C^T \otimes (A^T J_{q+1,2q}) \\ &\qquad +I_{q+1} \otimes (q+1)J_{2q} \\ &= (J_{q+1} - I_{q+1}) \otimes (E^T J_q E) + I_{q+1} \otimes qE^T E + qI_{q+1} \otimes (A^T A) \\ &\qquad\qquad +I_{q+1} \otimes (q+1)J_{2q} \\ &= (J_{q+1} - I_{q+1}) \otimes (-E^T J_{q,2q}) + qI_{q+1} \otimes (E^T E + A^T A) \\ &\qquad +(q+1)I_q \otimes J_{2q} \\ &= (J_{q+1} - I_{q+1}) \otimes J_{2q} + qI_{q+1} \otimes ((2q+2)I_{2q} - J_{2q}) \\ &\qquad +(q+1)I_q \otimes J_{2q} \\ &= qI_{q+1} \otimes ((2q+2)I_{2q} - J_{2q}) + (qI_{q+1} + J_{q+1}) \otimes J_{2q} \\ &= (2q^2+2q)I_{2q^2+2q} + J_{2q^2+2q}. \end{aligned}$$

From this it follows immediately that $N_q^T N_q = (2q^2+2q)I_{2q^2+2q+1} + J_{2q^2+2q+1}$ and thence that $N_q \in \mathcal{B}_{2q^2+2q+1}$.

4 The proof of Theorem C

Let $N \in \mathcal{B}_k$, $k = 2q^2 + 2q + 1$, be one of the examples constructed section 3, i.e. we may assume $NN^T = (k-1)I_k + J_k$. Recall that each row and each column of such an N has q^2 entries 1 and $(q+1)^2$ entries -1. Let

$$B \;=\; \begin{pmatrix} 1 & 1 ... 1 & 1 ... 1 \\ 1 & & \\ \vdots & N & N \\ 1 & & \\ -1 & & \\ \vdots & N & -N \\ -1 & & \end{pmatrix}. \tag{31}$$

Then $B \in \mathcal{H}_n$ where $n = 2k+1 = 4q^2 + 4q + 3$. We note that $n \equiv 3 \,(\mathrm{mod}\, 4)$. It is easy to see that

$$BB^T = \begin{pmatrix} n & -4q-1 \cdots -4q-1 & -1 ... -1 \\ -4q-1 & & \\ \vdots & (n-3)I_k + 3J_k & -J_k \\ -4q-1 & & \\ -1 & & \\ \vdots & -J_k & (n-3)I_k + 3J_k \\ -1 & & \end{pmatrix}. \tag{32}$$

Let

$$h_n = \sqrt{\frac{4 \cdot 11^6}{7^7}(n-3)^{n-7} n^7} \tag{33}$$

be the bound in (5) of Proposition C.

Lemma 4.1 *Assume that B is as in (31). Then*

1.

$$d_n = det BB^T = \\ (n - y - kt(4q + y + 1))(n + k(3 - y) - 2)(n + 3k - 2)(n-3)^{n-3},$$

where

$$\begin{aligned} y &= \frac{k}{5(k-1)} \text{ and} \\ t &= \frac{4q+y+1}{n+k(3-y)-3} \end{aligned}$$

2.

$$\lim_{n\to\infty} \frac{det B}{\sqrt{n^n}} = \sqrt{e^{-3}}.$$

3.

$$\lim_{n\to\infty} \sqrt{\frac{d_n}{h_n}} = \frac{1}{2}\left(\frac{7}{11}\right)^3 \sqrt{7} \approx 0.34.$$

Proof

1. The formula for d_n is obtained by first zeroing out the $-J$ matrix in the upper right hand corner of BB^T and secondly zeroing out the first row of the matrix that remains in the upper left hand corner.

2. The critical term in the formula for d_n occurs in the first factor. We see that

$$\begin{aligned} \lim_{q\to\infty} y &= 1/5 \text{ and} \\ \lim_{q\to\infty} qt &= 5/12 \end{aligned}$$

   The analysis of the remaining terms is straightforward.

3. This follows directly from part 2 of this lemma and (1.2) of [Eh2]

∎

Theorem C is now a consequence of lemma 4.1.

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