Abstract. The study of extremal problems related to independent sets in hypergraphs is a problem that has generated much interest. While independent sets in graphs are defined as sets of vertices containing no edges, hypergraphs have different types of independent sets depending on the number of vertices from an independent set allowed in an edge. We say that a subset of vertices is j-*independent* if its intersection with any edge has size strictly less than j. The Kruskal-Katona theorem shows that in an r-uniform hypergraph with a fixed size and order, the hypergraph with the most r-independent sets is the lexicographic hypergraph. In this paper, we use a hypergraph regularity lemma, along with a technique developed by Loh, Pikhurko, and Sudakov [9], to give an asymptotically best possible upper bound on the number of j-independent sets in an r-uniform hypergraph.

Key words. hypergraphs, independent sets, regularity lemma

Hypergraph independent sets

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1. Introduction

Independent sets in hypergraphs have been a topic of much interest. Much of this research has focused on determining algorithms for finding independent sets in r-uniform hypergraphs (see, e.g., [13]). Extremal questions related to maximizing the number of independent sets in graphs have also been well-studied. Kahn [6] solved the question among regular bipartite graphs and his theorem was recently extended to general regular graphs by Zhao [14]. When restricted to graphs of a fixed size and order, the result follows from the Kruskal-Katona theorem [8, 7] and the extremal graph is the lexicographic graph.

In this paper we determine asymptotically the maximum number of independent sets possible for an r-uniform hypergraph on n vertices and m edges. Independent sets in hypergraphs are a bit more complicated than those in graphs since they can be defined in a number of ways depending on how many vertices from an independent set are allowed in an edge.

Definition 1. *For an r-uniform hypergraph $\mathcal{H}$ and integers j with $1 \leq j \leq r$, let*

$$\mathcal{I}_j(\mathcal{H}) = \{S \subseteq V(\mathcal{H}) : |S \cap e| < j \text{ for all } e \in E(\mathcal{H})\}$$

and $i_j(\mathcal{H}) = |\mathcal{I}_j(\mathcal{H})|$.

Example 2. If $r = 2$ and $j = 2$, then $\mathcal{I}_2(G)$ is the collection of all independent sets in the graph G. For arbitrary r and $j = 1$, the set $\mathcal{I}_1(\mathcal{H})$ is simply the collection of subsets disjoint from all edges of $\mathcal{H}$. Thus, $i_1(\mathcal{H}) = 2^{n_0}$ where n_0 is the number of isolated vertices in $\mathcal{H}$.

In the case $j = r$, the maximum number of independent sets possible in an r-uniform hypergraph is known exactly as a consequence of the Kruskal-Katona theorem. For the sake of completeness, we include this result and its proof.

Theorem 3. *If $\mathcal{H} = (V, E)$ is an r-uniform hypergraph with $|V| = n$ and $|E| = m$, then*

$$i_r(\mathcal{H}) \leq i_r(\mathcal{L}_{n,m}),$$

where $\mathcal{L}_{n,m}$ *is the* r*-graph on* $[n]$ *whose edges are the first* m *elements in the lexicographic ordering on* $\binom{[n]}{r}$*.*

Proof. Let us write $\mathcal{I}_r^{(k)}(\mathcal{H})$ for $\mathcal{I}_r(\mathcal{H}) \cap \binom{V}{k}$. Note that $I \in \mathcal{I}_r^{(k)}(\mathcal{H})$ if and only if I is not in the upper shadow of $\mathcal{H}$ on level k, the set $\partial^{(k)}\mathcal{H} = \left\{B \in \binom{V}{k} : \exists\, e \in \mathcal{H} \text{ such that } e \subseteq B\right\}$. Thus

$$\begin{aligned} i_r(\mathcal{H}) &= \sum_{k=0}^{n} \left|\mathcal{I}_r^{(k)}(\mathcal{H})\right| \\ &= \sum_{k=0}^{r-1} \binom{n}{k} + \sum_{k=r}^{n} \left(\binom{n}{k} - \left|\partial^{(k)}\mathcal{H}\right|\right) \\ &\leq \sum_{k=0}^{r-1} \binom{n}{k} + \sum_{k=r}^{n} \left(\binom{n}{k} - \left|\partial^{(k)}\mathcal{L}_{n,m}\right|\right) \\ &= i_r(\mathcal{L}_{n,m}), \end{aligned}$$

where the inequality $\left|\partial^{(k)}\mathcal{H}\right| \geq \left|\partial^{(k)}\mathcal{L}_{n,m}\right|$ is the content of the Kruskal-Katona theorem.

For $1 < j < r$, however, the problem of maximizing the number of j-independent sets in r-uniform hypergraphs seems to be open. We give an asymptotic answer to this question in terms of the number of independent sets in what we call a split hypergraph. Our approach follows that of Loh, Pikhurko and Sudakov in [9], where they determine asymptotically the maximum number of q-colorings of a graph G with n vertices and m edges. They use Szemerédi's Regularity Lemma [11] in a clever way; given a regular partition of the vertex set and and q-coloring of G, they associate a form of auxiliary coloring of the auxiliary graph. Since the regular partition has a bounded number of parts, there are only a constant number of possible auxiliary colorings and one need only consider, asymptotically, those colorings of G corresponding to a fixed auxiliary coloring. This allows them to get good control on the problem of maximizing the number of q-colorings.

We adapt the approach to prove an asymptotic bound on the number of independent sets in an r-uniform hypergraph. Of course, in doing this, we need to use a hypergraph regularity lemma, which we present in Section 2. Also in that section, we present some lemmas related to the asymptotic extremal graphs, which we call *split hypergraphs*. They are defined as follows.

Definition 4. *The* j-split r-graph with partition (A, B)*, denoted* $\mathcal{S}_j^{(r)}(A, B)$*, is defined as the* r*-uniform hypergraph with vertex set* $V = A \cup B$ *and edge set*

$$\left\{e \in \binom{V}{r} : |e \cap A| < j\right\}.$$

When we are not concerned with the identity of the sets A *and* B*, we write* $\mathcal{S}_j^{(r)}(k, n-k)$ *for a* j*-split* r*-graph with* $|A| = k$ *and* $|B| = n - k$*.*

We can now state the main result of the paper.

Main Theorem. *Given $\eta > 0$ and any r-uniform hypergraph $\mathcal{H}$ on n vertices with $\eta\binom{n}{r} \leq e(\mathcal{H}) \leq (1-\eta)\binom{n}{r}$, if we let k^* be maximal such that*

$$e(\mathcal{H}) \leq \left|\mathcal{S}_j^{(r)}(k^*, n-k^*)\right|,$$

then

$$\log_2 i_j(\mathcal{H}) \leq (1+o(1))k^*.$$

In Section 3, we prove this theorem.

2. Some preliminaries

There are now highly sophisticated hypergraph regularity lemmas available. (See, e.g., [1,4, 5,10,12].) We, however, need only a simple version which can be found, e.g., in [2] or [3]. In order to state the regularity lemma, we need first to define an ε-regular partition, the structure guaranteed by the regularity lemma. In what follows, we use the standard notation that if $\mathcal{H}$ is an r-uniform hypergraph and $W_1, W_2, \ldots, W_r$ are disjoint subsets of the vertex set, then

$$\mathcal{H}[W_1, W_2, \ldots, W_r] = \{e \in \mathcal{H} \,:\, |e \cap W_i| = 1, i = 1, 2, \ldots, r\}.$$

Definition 5. *Let $\mathcal{H} = (V, E)$ be an r-uniform hypergraph. Given $\varepsilon > 0$, we say an r-tuple $(W_1, W_2, \ldots, W_r)$ of disjoint subsets of V is* ε-regular *if for all subsets $S_i \subset W_i$ with $|S_i| \geq \varepsilon\,|W_i|$, we have*

$$\left| \frac{|\mathcal{H}[W_1, W_2, \ldots, W_r]|}{\prod_1^r |W_i|} - \frac{|\mathcal{H}[S_1, S_2, \ldots, S_r]|}{\prod_1^r |S_i|} \right| < \varepsilon.$$

We say a partition $\{V_1, \ldots, V_t\}$ is an ε-regular partition of $\mathcal{H}$ *if*
1) $|V_1| \leq |V_2| \leq \cdots \leq |V_t| \leq |V_1| + 1$, and
2) the r-tuple $(V_{i_1}, V_{i_2}, \ldots, V_{i_r})$ is ε-regular for all but εt^r sequences $(i_1, i_2, \ldots, i_r) \in [t]^r$.

The following hypergraph regularity lemma can be read out of, for example, a result of Czygrinow and Rödl [2].

Theorem 6. *For all $r, m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $M, L \in \mathbb{N}$ such that given any r-uniform hypergraph $\mathcal{H} = (V, E)$ with $|V| \geq L$, there is an ε-regular partition $\{V_1, \ldots, V_t\}$ of $\mathcal{H}$ with $m \leq t \leq M$.*

We now present some characteristics of the asymptotic extremal graphs, $\mathcal{S}_j^{(r)}(k, n-k)$, which were defined in Section 1. Note that in the case when $r = 2$, only two values for j are of interest, namely 1 and 2. (If $j \geq 3$, then $\mathcal{S}_j^{(2)}(k, n-k)$ is a complete graph.) In the case when $j = 1$, we have that $\mathcal{S}_1^{(2)}(k, n-k)$ is the disjoint union of E_k and K_{n-k}. When $j = 2$, similarly, $\mathcal{S}_2^{(2)}(k, n-k)$ is the join of E_k and K_{n-k}. The following lemma gives the number of edges and j-independent sets in the split r-graphs. We write

$$\binom{n}{\leq k} = \sum_{i=0}^{k} \binom{n}{i}.$$

Lemma 7. *Let n and r be positive integers with $n \geq r$. If $|A| = k$ and $|B| = n-k$, then the number of edges in $\mathcal{S} = \mathcal{S}_j^{(r)}(A, B)$ is*

$$e(\mathcal{S}) = \sum_{i=0}^{j-1} \binom{k}{i}\binom{n-k}{r-i}.$$

If $k < j-1$, then $\mathcal{S}$ is complete, and so $i_j(\mathcal{S}) = \binom{n}{\leq j-1}$. If $n < k+r-j+1$, then $\mathcal{S}$ is empty, so $i_j(\mathcal{S}) = 2^n$. Otherwise, i.e., when $k \geq j-1$ and $n \geq k+r-j+1$, the number of j-independent sets in $\mathcal{S}$ is

$$i_j(\mathcal{S}) = 2^k + \binom{n}{\leq j-1} - \binom{k}{\leq j-1}.$$

Proof. The first calculation is straightforward. For the second, note that any subset of A is j-independent, as is any subset of V of size at most $j-1$. On the other hand, if I is of size at least j and is not contained in A, we claim that I is not j-independent. We first show that if $|I| = j$ and $I \not\subseteq A$, then I is not j-independent. From this, the more general statement follows by considering any j-subset of I not contained in A.

Let $c = |I \cap A|$. If $r \leq n-k+c$, then there is enough room to complete I to an edge using only elements from B. Pick an arbitrary $(r-j)$-subset e' of $B \setminus I$ and let $e = I \cup e'$. Then e has size r, contains I, and $|e \cap A| = |I \cap A| = c < j$, and so $e \in \mathcal{S}$. On the other hand, if $n-k+c < r \leq n-k+j-1$, choose an arbitrary $(r+k-n-c)$-subset e'' of $A \setminus I$ and set $e = B \cup I \cup e''$. Then e has size r, contains I, and

$$|e \cap A| = |I \cap A| + |e''| = c + r + k - n - c = r + k - n \leq j - 1,$$

and hence $e \in \mathcal{S}$.

The next lemma bounds the difference in the number of edges between split graphs with adjacent values of k. Although officially a hypergraph $\mathcal{H}$ is a pair consisting of a vertex set V and an edge set contained in $\mathcal{P}(V)$, in most cases from this point we suppress the vertex set, and so, for instance, we write $|\mathcal{H}|$ for the number of edges in $\mathcal{H}$.

Lemma 8. *Given $0 < \xi < 1/2$ there exists $\zeta > 0$ such that whenever $k \in [n]$ and $\frac{k}{n} \in (\xi, 1-\xi)$ and $n \geq 2\max(j, r-j)/\xi$, we have*

$$\left|\mathcal{S}_j^{(r)}(k-1, n-k+1)\right| \geq \left|\mathcal{S}_j^{(r)}(k, n-k)\right| + \zeta n^{r-1}.$$

Proof. Writing $\mathcal{S} = \mathcal{S}_j^{(r)}(k-1, n-k+1)$ and $\mathcal{S}' = \mathcal{S}_j^{(r)}(k, n-k)$, we see that $\mathcal{S}$ contains $\mathcal{S}'$ and

$$\begin{aligned}
|\mathcal{S} \setminus \mathcal{S}'| &= \binom{k-1}{j-1}\binom{n-k}{r-j} \\
&\geq \frac{(k-j)^{j-1}}{(j-1)!} \cdot \frac{(n-k-r+j)^{r-j}}{(r-j)!} \\
&\geq \frac{1}{(j-1)!(r-j)!}\left(\frac{\xi\, n}{2}\right)^{j-1}\left(\frac{\xi\, n}{2}\right)^{r-j} \\
&= \frac{\xi^{r-1}}{2^{r-1}(j-1)!(r-j)!} \cdot n^{r-1}.
\end{aligned}$$

We can clearly set $\zeta = \xi^{r-1}/\left(2^{r-1}(j-1)!(r-j)!\right)$.

Our last lemma discusses the relationship between the number of edges in the split hypergraph $\mathcal{S}_j^{(r)}(k, n-k)$ and the ratio k/n. To be more precise we will define, for given n, j, r, and e,

$$\begin{aligned}
s(k) &= e(\mathcal{S}_j^{(r)}(k, n-k)) \\
k^* &= \max\left\{k \leq n \,:\, s(k) \geq e\right\}.
\end{aligned}$$

Our lemma shows that if $e\big/\binom{n}{r}$ is bounded away from 0 and 1 then so is k^*/n.

Lemma 9. *Given $\eta > 0$ and a positive integer r, there exists $\xi = \xi(\varepsilon, r) \in (0, 1/2)$ such that for n sufficiently large (as a function of η and r), and any positive integer $j \leq r$, we have the following: if e satisfies*

$$\eta\binom{n}{r} < e < (1-\eta)\binom{n}{r},$$

then k^* satisfies

$$\xi < \frac{k^*}{n} < 1 - \xi.$$

Proof. For the lower bound (by Lemma 7), we want to show that there is a ξ such that

$$s(\xi n) = \sum_{i=0}^{j-1}\binom{\xi n}{i}\binom{n-\xi n}{r-i} \geq (1-\eta)\binom{n}{r}.$$

Pick ξ_1 with $0 < \xi_1 < 1 - (1-\eta)^{1/r}$. To show the above with $\xi = \xi_1$, we bound the sum by the $i = 0$ term. Thus,

$$\begin{aligned} s(\xi_1 n) &= \sum_{i=0}^{j-1} \binom{\xi_1 n}{i}\binom{(1-\xi_1)n}{r-i} \\ &\geq \binom{(1-\xi_1)n}{r} \\ &\geq (1-o(1))(1-\xi_1)^r \binom{n}{r} \\ &> (1-\eta)\binom{n}{r}, \end{aligned}$$

for n sufficiently large. (The $o(1)$ term is uniform in $\xi_1 \in (0, 1/2)$.) In the other direction note that if $0 < \xi_2 < \eta / \left(j\binom{r}{\lfloor r/2 \rfloor}\right)$, and we let i' be the i with $\binom{(1-\xi_2)n}{i}\binom{\xi_2 n}{r-i}$ maximum, then

$$\begin{aligned} s((1-\xi_2)n) &= \sum_{i=0}^{j-1} \binom{(1-\xi_2)n}{i}\binom{\xi_2 n}{r-i} \\ &\leq j\binom{(1-\xi_2)n}{i'}\binom{\xi_2 n}{r-i'} \\ &< j(1-\xi_2)^{i'}\xi_2^{r-i'}\binom{n}{i'}\binom{n}{r-i'} \\ &< j(1-\xi_2)^{i'}\xi_2^{r-i'}(1+o(1))\binom{n}{i'}\binom{n-i'}{r-i'} \\ &< j(1-\xi_2)^{i'}\xi_2^{r-i'}(1+o(1))\binom{n}{r}\binom{r}{i'} \\ &< j\xi_2(1+o(1))\binom{n}{r}\binom{r}{\lfloor r/2 \rfloor} \\ &< \eta\binom{n}{r}, \end{aligned}$$

for n sufficiently large. Letting $\xi = \min(\xi_1, \xi_2)$, the lemma follows.

3. Proof of Main Theorem

We restate the main theorem before presenting its proof.

Main Theorem. *Given $\eta > 0$ and any r-uniform hypergraph $\mathcal{H}$ on n vertices with $\eta\binom{n}{r} \leq e(\mathcal{H}) \leq (1-\eta)\binom{n}{r}$, if we let k^* be maximal such that*

$$e(\mathcal{H}) \leq \left|\mathcal{S}_j^{(r)}(k^*, n-k^*)\right|,$$

then

$$\log_2 i_j(\mathcal{H}) \le (1+o(1))k^*.$$

Proof. Our proof proceeds in a sequence of steps.

Step 1. Given $0<\delta<1$, there exists a "cleaned-up" subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ and a δ-regular partition $\{V_1, V_2, \ldots, V_t\}$ of $V(\mathcal{H})$ such that
a) $e(\mathcal{H}') \ge e(\mathcal{H}) - \delta n^r$,
b) all edges of $\mathcal{H}'$ span r parts of the partition, and
c) all subgraphs $\mathcal{H}'[V_{i_1}, V_{i_2}, \ldots, V_{i_r}]$ are either empty or δ-regular with density at least δ.

To show this, first apply Theorem 6 with $\varepsilon = \delta/4$ and m sufficiently large that

$$\frac{1}{r!}\left(1-\exp\left(\frac{-r(r-1)}{2m}\right)\right) < \delta/2$$

to get a suitable partition $\{V_1, V_2, \ldots, V_t\}$. We get $\mathcal{H}'$ by first deleting all edges that do not span r parts of the partition of which there at most

$$\begin{aligned}(1+o(1))\binom{n}{r}\left[1-\frac{n^r}{(n)_r}\prod_{i=1}^{r-1}\left(1-\frac{i}{t}\right)\right] &\le (1+o(1))\frac{n^r}{r!}\left(1-\exp\left(-\frac{r(r-1)}{2t}\right)\right)\\ &\le (1+o(1))\frac{n^r}{r!}\left(1-\exp\left(-\frac{r(r-1)}{2m}\right)\right)\\ &\le \frac{\delta}{2}n^r,\end{aligned}$$

for n sufficiently large. (In the in first inequality, we used the fact that $1-x \le e^{-x}$.) Now we additionally delete all edges from $\mathcal{H}[V_{i_1}, V_{i_2}, \ldots, V_{i_r}]$ whenever this either has density less than δ or is not δ-regular. The total number of such edges is at most

$$\varepsilon\binom{t}{r}\left(\frac{n}{t}\right)^r + \varepsilon t^r\left(\frac{n}{t}\right)^r \le 2\varepsilon n^r \le \frac{\delta}{2}n^r.$$

$\mathcal{H}'$ is the result of deleting all of the above edges from $\mathcal{H}$.

Definition 10. *Given a j-independent set I in $\mathcal{H}'$ we define a subset $\mathcal{D}(I) \subseteq [t]$ by*

$$\mathcal{D}(I) = \{i \in [t] \,:\, |I \cap V_i| \ge \delta\,|V_i|\}\,.$$

We call a j-independent set robust *if $I \cap V_i = \emptyset$ for all $i \notin \mathcal{D}(I)$. For $D \subseteq [t]$, define*

$$V_D = \bigcup_{i\in D} V_i.$$

Step 2. The subset $\mathcal{R} \subseteq \mathcal{I}_j(\mathcal{H}')$ consisting of robust j-independent sets I has size at least $\exp(-c_\delta n) i_j(\mathcal{H}')$, where c_δ tends to zero as δ tends to zero.

In order to prove this, define a map $f : \mathcal{I}_j(\mathcal{H}') \to \mathcal{R}$ by

$$f(I) = I \cap V_{\mathcal{D}(I)}.$$

We show that each $I_0 \in \mathcal{R}$ has at most $\exp(c_\delta n)$ preimages under f. First note that $|I \setminus f(I)| \le \delta n$, therefore the number of I such that $f(I) = I_0$ is at most

$$\begin{aligned}\sum_{s=0}^{\delta n} \binom{n}{s} &\le (1+\delta n)\left(\frac{en}{\delta n}\right)^{\delta n} \\ &\le e^{\delta n}\left(\frac{e}{\delta}\right)^{\delta n} \\ &= e^{(2\delta - \delta \ln \delta)n}.\end{aligned}$$

Setting $c_\delta = 2\delta - \delta \ln \delta$ proves the claim.

Step 3. Let $\mathcal{S}_D$ be the j-split hypergraph $\mathcal{S}_j^{(r)}(V_D, V_D^c)$. If there exists $I \in \mathcal{R}$ such that $\mathcal{D}(I) = D$, then $\mathcal{H}' \subseteq \mathcal{S}_D$.

We need to show that if $\{V_{i_1}, V_{i_2}, \ldots, V_{i_r}\}$ is a set of blocks with $|\{i_1, i_2, \ldots, i_r\} \cap D| \ge j$, then $\mathcal{H}'[V_{i_1}, V_{i_2}, \ldots, V_{i_r}]$ is empty. If it is not empty then it is δ-regular. For $i \in \{i_1, i_2, \ldots, i_r\} \cap D$, we have $|I \cap V_i| \ge \delta |V_i|$, so, by the δ-regularity, there is an edge $e \in \mathcal{H}'$ with $e \cap V_i \subseteq I \cap V_i$ for all $i \in \{i_1, i_2, \ldots, i_r\} \cap D$. In particular, $|e \cap I| \ge j$, contradicting the j-independence of I.

Step 4. Finally, we prove the conclusion of the Main Theorem, that

$$\log_2(i_j(\mathcal{H})) \le (1+\eta)k^*.$$

For $D \subseteq [t]$, define $\mathcal{R}_D = \{I \in \mathcal{R} : \mathcal{D}(I) = D\}$. Fix an D^* with $|\mathcal{R}_{D^*}|$ maximal, so that

$$|\mathcal{R}_{D^*}| \ge 2^{-t}|\mathcal{R}| \ge 2^{-M}|\mathcal{R}|. \qquad (\dagger)$$

We need to bound $|V_{D^*}|$. Since $|\mathcal{H}| \le |\mathcal{S}_{D^*}| + \delta n^r$, by Lemma 8, $|V_{D^*}| \le k^* + c'_\delta n$. Here $c'_\delta = \delta/\zeta$ tends to zero as $\delta \to 0$. Then

$$\begin{aligned}i_j(\mathcal{H}) &\le i_j(\mathcal{H}') \\ &\le e^{c_\delta n}|\mathcal{R}| \\ &\le e^{c_\delta n} 2^M |\mathcal{R}_{D^*}| \\ &\le e^{c_\delta n} 2^M 2^{|V_{D^*}|}.\end{aligned}$$

The first inequality follows from the fact that $\mathcal{H}' \subseteq \mathcal{H}$, the second from Step 2, and the third from ($\dagger$). The final estimate is simply the fact that if I is a robust independent set with $\mathcal{D}(I) = D$, then $I \subseteq V_D$. Taking logarithms, we have

$$\begin{aligned}\log_2(i_j(\mathcal{H})) &\le \frac{c_\delta n}{\ln 2} + M + |V_{D^*}| \\ &\le \frac{c_\delta n}{\ln 2} + M + k^* + c'_\delta n \\ &\le k^* + \left(\frac{c_\delta}{\ln 2} + c'_\delta\right)\xi k^* + M \\ &\le (1+o(1))k^*,\end{aligned}$$

where the penultimate step follows from Lemma 9.

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