Reconstructing under group actions

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Abstract. We give a bound on the reconstructibility of an action $G \rightarrowtail X$ in terms of the reconstructibility of a the action $N \rightarrowtail X$, where N is a normal subgroup of G, and the reconstructibility of the quotient G/N. We also show that if the action $G \rightarrowtail X$ is *locally finite*, in the sense that every point is either in an orbit by itself or has finite stabilizer, then the reconstructibility of $G \rightarrowtail X$ is at most the reconstructibility of G. Finally, we give some applications to geometric reconstruction problems.

Key words. Insert your keywords here.

1. Introduction

Combinatorial reconstruction problems arise when we are given the sub-objects of a certain size of some combinatorial object, up to isomorphism, and are asked whether this is sufficient information to reconstruct the original object. For instance the famous Reconstruction Conjecture, made sixty years ago by S.M. Ulam and P.J. Kelly (see [10] and [29]), asserts that every finite graph on $n \geq 3$ vertices can be reconstructed from the collection of all its (non-trivial) induced subgraphs. Similarly the Edge Reconstruction Conjecture (Harary [9]) asserts that a graph with $m \geq 4$ edges can be reconstructed from the collection of all its (non-trivial) subgraphs. There is a substantial literature on graph reconstruction (see, for instance, [2], [1], [3], [19]). Reconstruction problems have been considered for a variety of combinatorial objects, including directed graphs [27,28], hypergraphs [12], infinite graphs [20], codes [14], sets of real numbers [24], sequences [26, 13], and combinatorial geometries [4,5].

The general setting for a reconstruction problem requires a notion of isomorphism and a notion of sub-object. Significant progress on this problem has been made in recent years ([1,6,8,7,16–18,15,25]) in the case where we have a group action $G \rightarrowtail X$ providing the notion of isomorphism, and we wish to reconstruct subsets of X from the multiset of isomorphism classes of their k-element subsets. This collection is called the *k-deck*.

Our aim in this paper is to prove some results about reconstruction in this context. We begin with two introductory sections: section 2 gives some definitions and examines several different ways in which the information contained in the k-deck can be presented, while section 3 develops the method of features (this was also a main tool in [21]).

In section 4, we address the question of which actions of a fixed group G are the hardest to reconstruct. We prove that the (left) regular action of G on itself is the hardest to reconstruct among all locally finite actions: in particular, if G is finite then the reconstructibility of any action of G is at most the reconstructibility of the regular action.

In section 5, we consider the relationship between the reconstructibilities of a group G acting on a set X and the action of a normal subgroup $N \lhd G$. We show that, provided the action of G/N on N-isomorphism classes in X is locally finite, the reconstructibility of the action of G is at most the product of the reconstructibility of the action of N and the reconstructibility of the regular action of G/N. As a corollary, we show that, for groups G and H, the reconstructibility of $G \times H$ is at most the product of the reconstructibilities of G and H.

In the final section, we give some applications to geometrical reconstruction problems.

2. Decks and Definitions

In [23] the current authors introduced some new techniques to the problem of reconstructing subsets under group action. In that paper we considered subsets of the cyclic group $\mathbb{Z}_n$ under the natural action of $\mathbb{Z}_n$ on itself. Our techniques involved considering the group ring $\mathbb{Q}\mathbb{Z}_n$, and extending the notions of deck and reconstructibility to elements of $\mathbb{Q}\,\mathbb{Z}_n$. Thus a subset of $\mathbb{Z}_n$ was regarded as merely a special case of a rational-valued weight function on $\mathbb{Z}_n$. A multiset in the group can (of course) also be regarded as an element of the group ring, and there is little difference between the problem of reconstructing a general rational-valued weight function on $\mathbb{Z}_n$ and that of reconstructing a general multiset in $\mathbb{Z}_n$. A Fourier-analytic approach was used in [23] to give bounds on the reconstructibility of rational-valued functions; this approach was extended by Pebody [22], who determined the reconstructibility of all finite Abelian groups, showing in particular that the reconstructibility of $\mathbb{Z}_n$ acting on itself is at most 6. This was used in [21] to show that the reconstructibility of finite multisets of $\mathbb{R}^2$ under the action of the group of rigid motions is finite.

Since we make use of some of the same techniques as [21] in the current paper we present in this section a careful discussion of our conventions regarding multisets, and a careful analysis of the relationship between the k-deck in its traditional form, and the extended notion presented in [23].

2.1. Multisets

We begin by discussing some of the conventions we adopt with respect to multisets.

Definition 1. *A* multiset S in a ground set X *is formally a function*

$$m_S : X \to \mathbb{N}$$

recording, for each element of X, the number of times it appears in S. The multiplicity *of x as an element of the multiset S is $m_S(x)$ and the* size $|K|$ *of a multiset K is the sum of the multiplicities of its elements. This might be infinite, though we will frequently restrict our attention to finite multisets in X. The* support *of a multiset S in X is the set* $\mathrm{supp}(S) = \{x \in X \,:\, m_S(x) > 0\}$. *We write $\mathcal{M}(X)$ for the collection of all finite multisets in X.*

Definition 2. *We say that a multiset K* is contained in a multiset S *if $m_X(x) \leq m_S(x)$ for every $x \in X$. We shall abuse terminology slightly and talk about the collection of all subsets of S, when we in fact mean the collection of all multisets contained in S. We write $\mathcal{P}(S)$ for this collection. It is important to notice that $\mathcal{P}(S)$ is itself a multiset, typically containing various sub-multisets of S multiple times: we adopt the convention that the multiplicity of $T \subset S$ in $\mathcal{P}(S)$ is given by*

$$m_{\mathcal{P}(S)}(T) = \prod_{x \in \mathrm{supp}(T)} \binom{m_S(x)}{m_T(x)}.$$

With this convention we have $|\mathcal{P}(S)| = 2^n$ whenever $|S| = n$.

At times we wish to only discuss multisets of a certain size. We therefore define the multisets

$$\begin{aligned} \mathcal{P}^{(\leq k)}(S) &= \{T \in \mathcal{P}(S) \,:\, |T| \leq k\} \\ \mathcal{M}^{(\leq k)}(X) &= \{S \in \mathcal{M}(X) \,:\, |S| \leq k\} \\ \mathcal{P}^{(k)}(S) &= \{T \in \mathcal{P}(S) \,:\, |T| = k\} \\ \mathcal{M}^{(k)}(X) &= \{S \in \mathcal{M}(X) \,:\, |S| = k\}. \end{aligned}$$

Definition 3. *When we have multisets in the picture we also have to consider two different, and equally natural, notions of union. The* multiset union *of a collection $\mathcal{S}$ of multisets (or sets) is the multiset $\bigoplus_{S \in \mathcal{S}} S$ in which each element has multiplicity equal to the sum of its multiplicities in each multiset in $\mathcal{S}$. The* set union *$\bigcup_{S \in \mathcal{S}} S$ gives to each element the maximum multiplicity with which it appears in any element of $\mathcal{S}$.*

2.2. Reconstruction and symmetry

In the following we suppose that a group action $G \rightarrowtail X$ has been specified. We write the group action generically as $(g, x) \mapsto g.x$. We first fix some standard notation from the theory of group actions, and then discuss the definitions relevant to reconstruction. Clearly the action on X naturally induces an action on $\mathcal{M}(X)$ by $g.S = \{g.x \,:\, x \in S\}$. Written in terms of multiplicities this states that $m_{g.S}(x) = m_S(g^{-1}.x)$.

Definition 4. *If K is a finite multiset in X we write* $\mathrm{Fix}(K)$ *for the subgroup of G which fixes K pointwise, and* $\mathrm{Stab}(K)$ *for the subgroup fixing K as a multiset. In other words*

$$\begin{aligned} \mathrm{Fix}(K) &= \{g \in G \,:\, g.x = x, \forall x \in K\} \\ \mathrm{Stab}(K) &= \{g \in G \,:\, g.K = K\}. \end{aligned}$$

Definition 5. *Given two multisets S, T in X we say that they are* isomorphic*, and write $S \simeq T$, if there exists $g \in G$ such that $g.S = T$. The collection of all multisets in X isomorphic to S is the* isomorphism class *of S, written $[S]_G$ (or simply $[S]$ if the group action is sufficiently clear).*

Definition 6. *If S is a multiset in X then the k*-deck *of S is the multiset $D_k(S) = \{[K]_G \,:\, K \in \mathcal{P}(S), |K| \leq k\}$. Note that if S is finite then*

$$|D_k(S)| = |\mathcal{P}^{(\leq k)}(S)| = \sum_{i=0}^{k} \binom{|S|}{i}.$$

We write $m_S([K]) = |\{L \in \mathcal{P}(S) : L \simeq K\}|$ *for the multiplicity of* $[K]$ *in* $D_k(S)$ *(where* $|K| \leq k$*). Thus*

$$m_S([K]) = \sum_{L \simeq K} m_S(L).$$

If we want to emphasize the particular group acting, we will write $D_k(G \rightarrowtail S)$*.*

Definition 7. *We say that a multiset* S *in* X *is* reconstructible from its k-deck *(or* k-reconstructible*) if every* $T \subset X$ *with* $D_k(T) = D_k(S)$ *is isomorphic to* S*. Similarly, if* $f : \mathcal{M}(X) \to Y$ *is an arbitrary function then we say* $f(S)$ *is* k-reconstructible *if* $D_k(T) = D_k(S) \Rightarrow f(T) = f(S)$*. More generally we say that* $f : \mathcal{M}(X) \to Y$ *is* k-reconstructible *if* $f(S)$ *is* k*-reconstructible for all* $S \subset X$*. This is equivalent to saying that* f *factors through the map* $S \mapsto D_k(S)$*. Note that if* f *is* k*-reconstructible it must depend only on* $[S]_G$*, since* $D_k(S)$ *depends only on* $[S]_G$*. We will say that* multisets in X are reconstructible from their k-decks *if* $S \mapsto [S]_G$ *is* k*-reconstructible. Sometimes we will be concerned with reconstruction based on data other than the* k*-deck. We say that one function* f *is reconstructible from another if* f *factors through* g*, and that* f *and* g *are* mutually reconstructible *if each factors through the other. Thus* f *and* g *are mutually reconstructible if* $f(S) = f(T)$ *iff* $g(S) = g(T)$*.*

Definition 8. *We define the* reconstructibility*,* $r_{\mathbb{N}}(G \rightarrowtail X)$*, of a group action* $G \rightarrowtail X$ *to be the minimum* k *such that all finite multisets in* X *are reconstructible from their* k*-decks. We write* $r_{\mathbb{N}}(G)$ *for the reconstructibility of the (left)-regular action* $G \rightarrowtail G$*.*

Remark 1. It is more traditional to define the k-deck in terms of the multiset $\{[K]_G : K \in \mathcal{P}^{(k)}(S)\}$ of subsets of S of size exactly k. We shall refer to this as the *strict k-deck of S*. However this leads to problems if $|S| < k$, since all sets of size less than k contain exactly the same collection of isomorphism classes of subsets of size k, viz. the empty collection. We use the definition above in order to make the statement of our results clearer. It is important to note that, by the same argument as in Kelly's lemma [11], the k-deck as we define it is reconstructible from the stricter notion, provided $k \leq |S| < \infty$.

Lemma 1. *If* S *is a finite multiset in* X *and* $|S| \geq k$ *then the* k*-deck of* S *is reconstructible from the strict* k*-deck.*

Proof. We prove first that the strict $(k-1)$-deck of S is reconstructible from the strict k-deck and proceed by induction. Consider then a fixed multiset L_0 in X of size $k-1$. We will do a double sum over pairs L, K where $L \simeq L_0$ is an element of the support of $\mathcal{P}^{(k-1)}(S)$ and K is an element of $\mathcal{P}^{(k)}(S)$ that contains L. Then

$$\begin{aligned}
m_S([L_0])(|S| - k + 1) &= \sum_{L \in \mathrm{Orb}(L_0)} \sum_{x \in X} m_S(L)(m_S(x) - m_L(x)) \\
&= \sum_{L \in \mathrm{Orb}(L_0)} \sum_{\substack{K \in \mathcal{M}^{(k)}(X) \\ K = L \oplus \{x\}}} m_S(L)(m_S(x) - m_L(x)) \\
&= \sum_{K \in \mathcal{M}^{(k)}(X)} \sum_{\substack{L \in \mathrm{Orb}(L_0) \\ L \subset K, K = L \oplus \{x\}}} (m_L(x) + 1) m_S(K) \\
&= \sum_{K \in \mathcal{M}^{(k)}(X)} m_K([L_0]) m_S(K).
\end{aligned}$$

Note that the third equality is true because if $K = L \oplus \{x\}$ then

$$\frac{m_S(K)}{m_S(L)} = \frac{m_S(x) - m_L(x)}{m_L(x) + 1}.$$

Since $m_K([L_0])$ is an isomorphism invariant of K, we are able to reconstruct $\sum_K m_K([L_0]) m_S(K)$ and hence $m_S([L_0])$ from the strict k-deck of S.

2.3. Different notions of deck

Instead of working directly with the k-deck of a multiset, it is often convenient to work with a reconstructible function that encodes the same information in a different form (see, e.g., [13, 26]). In this section we discuss the relationship between several ways in which the deck can be presented. We will suppose for the remainder of the section that we have a group action $G \rightarrowtail X$ and a multiset S in X. We are concerned with the following objects.

- The function $d_{S,k} : \mathcal{M}^{(\leq k)}(X) \to \mathbb{N} \cup \{\infty\}$ defined by

$$d_{S,k}(K) = \sum_{g \in G} \prod_{x \in K} m_S(g.x) = \sum_{g \in G} \prod_{y \in \mathrm{supp}(g.K)} m_S(y)^{m_{g.K}(y)}$$

- The function $\tilde{d}_{S,k} : \mathcal{M}^{(\leq k)}(X) \to \mathbb{N} \cup \{\infty\}$ defined by

$$\tilde{d}_{S,k}(K) = \sum_{L \in \mathrm{Orb}(K)} \prod_{x \in L} m_S(x) = \sum_{L \in \mathrm{Orb}(K)} \prod_{x \in \mathrm{supp}(L)} m_S(x)^{m_L(x)}.$$

- The k-deck of S, that is the multiset

$$D_k(S) = \{[K]_G \,:\, K \in \mathcal{P}(S), |K| \leq k\},$$

  or equivalently the function m_S on $\mathcal{M}^{(\leq k)}(X)$ giving, for any K, the multiplicity of $[K]$ in $D_k(S)$.

These three functions are very closely related, and in [23] we adopted $d_{S,k}$ as our notion of deck. The other two functions arise from different ways of generalizing the notion of deck to cover multisets. and can be thought of as the deck "with or without replacement". Note that if all the multiplicities of elements of S are 1 then $\tilde{d}_{S,k}(K) = m_S([K])$, so that $\tilde{d}_{S,k}$ and $D_k(S)$ are equivalent.

We prove in this section that $D_k(S)$ and $\tilde{d}_{S,k}$ give identical information about S, and that $\tilde{d}_{S,k}$ and $d_{S,k}$ are mutually reconstructible under suitable finiteness conditions. In particular, if S and G are both finite then all three notions are mutually reconstructible.

Our setting will involve a finite ground set A acted on by a finite group H. Given a multiset K in X we let $A = \mathrm{supp}(K)$ and define polynomials p_K, g_K over variables $(z_a)_{a \in A}$ by

$$p_K = \prod_{a \in A} \binom{z_a}{m_K(a)}$$

$$q_K = \prod_{a \in A} z_a^{m_K(a)}.$$

It is clear that the polynomial p_K is a linear combination of the collection $\{q_L : A \subset L \subset K\}$ and conversely. In other words, there exist rationals $\lambda_{K,L}$ and $\mu_{K,L}$ such that

$$\begin{aligned} p_K &= \sum_{A\subset L\subset K} \lambda_{K,L} q_L \\ q_K &= \sum_{A\subset L\subset K} \mu_{K,L} p_L. \end{aligned}$$

Lemma 2. *For all k the functions $\tilde{d}_{S,k}$ and $D_k(S)$ of S are mutually reconstructible.*

Proof. Let S be a finite multiset in X. Set $A = \mathrm{supp}(S)$ and let T be a set of representatives for the left cosets of $\mathrm{Fix}(A)$. If K is a finite multiset of size k with $K \subset S$ we define, for $A \subset L \subset K$, $s(L)$ to be the number of left cosets of $\mathrm{Fix}(A)$ in $\mathrm{Stab}(L)$. (Note that $s(L) \le |\mathrm{supp}(K)|!$, so in particular $s(L)$ is finite.)

Then for $A \subset L \subset K$

$$\begin{aligned} m_S([L]) &= \sum_{L'\in \mathrm{Orb}(L)} m_{\mathcal{P}(S)}(L') \\ &= \frac{1}{s(L)} \sum_{g\in T} m_{\mathcal{P}(S)}(g.L) \\ &= \frac{1}{s(L)} \sum_{g\in T} \prod_{x\in A} \binom{m_S(g.x)}{m_L(x)} \end{aligned}$$

and

$$\begin{aligned} \tilde{d}_{S,k}(L) &= \sum_{L'\in \mathrm{Orb}(L)} \prod_{x\in \mathrm{supp}(L')} m_S(x)^{m_{L'}(x)} \\ &= \frac{1}{s(L)} \sum_{g\in T} \prod_{x\in A} m_S(g.x)^{m_L(x)}. \end{aligned}$$

So

$$\begin{aligned} m_S([K]) &= \frac{1}{s(K)} \sum_{g\in T} \sum_{A\subset L\subset K} \lambda_{K,L} \prod_{x\in A} m_S(g.x)^{m_L(x)} \\ &= \sum_{A\subset L\subset K} \lambda_{K,L} \frac{s(L)}{s(K)} \tilde{d}_{S,k}(L). \end{aligned}$$

Similarly,

$$\begin{aligned} \tilde{d}_{S,k}(K) &= \frac{1}{s(K)} \sum_{g\in T} \sum_{A\subset L\subset K} \mu_{K,L} \prod_{x\in A} \binom{m_S(g.x)}{m_L(x)} \\ &= \sum_{A\subset L\subset K} \mu_{K,L} \frac{s(L)}{s(K)} m_S([L]). \end{aligned}$$

Lemma 3. *If S is a multiset in X and $k \in \mathbb{N}$ then $\tilde{d}_{S,k}$ is reconstructible from $d_{S,k}$. If in addition points in X have finite stabilizers then $d_{S,k}$ and $\tilde{d}_{S,k}$ are mutually reconstructible.*

Proof. Given a multiset $K \in \mathcal{M}^{(\le k)}(X)$ we know that there is a canonical bijection between $G/\operatorname{Stab}(K)$ and $\operatorname{Orb}(K)$ by $g\operatorname{Stab}(K) \mapsto gK$. Let Y be a set of coset representatives for $\operatorname{Stab}(K)$, then

$$\begin{aligned} d_{S,k}(K) &= \sum_{g\in G} \prod_{x\in K} m_S(g.x) \\ &= \sum_{g\in Y} \sum_{h\in g\operatorname{Stab}(K)} \prod_{x\in K} m_S(h.x) \\ &= \sum_{L\in \operatorname{Orb}(K)} |\operatorname{Stab}(K)| \prod_{x\in L} m_S(x) \\ &= |\operatorname{Stab}(K)|\, \tilde{d}_{S,k}(K). \end{aligned}$$

Thus $\tilde{d}_{S,k}$ determines $d_{S,k}$. Clearly the same calculation proves that $d_{S,k}$ determines $\tilde{d}_{S,k}$ provided $|\operatorname{Stab}(K)|$ is finite.

3. The method of features

In [21] we proved a theorem, called there the Feature Theorem, which was extremely useful in that paper. Our current work also uses the Feature Theorem, and we include here, for completeness, a proof of this theorem. The Feature Theorem shows that from an appropriately sized deck of $G \rightarrowtail S$ we can reconstruct the k-deck of any collection of features naturally associated with configurations lying in S. To make this clearer let us give an example.

Example 1. We would like to associate to a configuration C in $\mathbb{R}^2$ a direction. This requires us to distinguish two points of C to use to define a reference line, whose direction we will call the *direction of* C. Thus we are led naturally to the notion of an oriented configuration in $\mathbb{R}^2$: an *oriented configuration in* $\mathbb{R}^2$ is a triple $\langle C, x, y\rangle$ consisting of a finite multiset C in $\mathbb{R}^2$ together with points $x, y \in \operatorname{supp}(C)$ with $x \neq y$. Given an oriented configuration $\langle C, x, y\rangle$ we can associate with it the "feature" $F(\langle C, x, y\rangle)$ defined to be the direction of the directed line segment from x to y. We consider this direction as an element of the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Notice that if R is the group of rigid motions of the plane and $\pi : R \to \mathbb{T}$ is the homomorphism which maps a rigid motion to its rotation angle, we have, for all $g \in R$,

$$F(g.\langle C, x, y\rangle) = \pi(g).F(\langle C, x, y\rangle).$$

With this example in mind we describe the general formalism we will use.

Definition 9. *A* configuration style *is a finite sequence* $a = (a_1, a_2, \ldots, a_r)$ *of positive integers. A* colored configuration in style a *is a pair* $\langle C, c\rangle$ *consisting of a finite multiset* C *in* X *and a coloring* $c : \operatorname{supp}(C) \to \{0, 1, \ldots, r\}$ *such that* $|c^{-1}(i)| = a_i$ *for* $i = 1, 2, \ldots, r$. *There is a natural action of* G *on colored configurations, where* $g.\langle C, c\rangle = \langle g.C, c \circ g^{-1}\rangle$. *Two colored configurations* $\langle C, c\rangle$ *and* $\langle C', c'\rangle$ *are therefore* isomorphic *if there exists* $g \in G$ *such that* $g.C = C'$ *and* $c'(g.x) = c(x)$ *for all* $x \in C$. *As usual we write* $[\langle C, c\rangle]_G$ *for the isomorphism class of* $\langle C, c\rangle$ *under the action of* G. *The* size *of a colored configuration* $\langle C, c\rangle$ *is simply the size of* C. *We write* $\mathcal{C}_a$ *for the collection of all colored configurations in style* a. *We say that* $\langle C, c\rangle$ *is an* a-colored configuration *in a multiset* S *if* c *is an* a*-coloring of* C *and* $C \subset S$.

Now we turn to the central reason for discussing colored configurations. We want to talk about a "feature" of a colored configuration, and, eventually, to be able to reconstruct the set of all such features associated with particular classes of configurations. Since these features are also the object of a reconstruction problem we insist that there be a group H acting on the features and that isomorphic colored configurations have isomorphic features.

Definition 10. *Given group actions $G \rightarrowtail X$ and $H \rightarrowtail Y$ we define an* H-feature *of a-colored configurations in X to be a function $f : \mathcal{C}_a \to Y$ on colored configurations together with a homomorphism $\phi : G \to H$ such that $f(g.\langle C, c\rangle) = \phi(g).f(\langle C, c\rangle)$ for all $\langle C, c\rangle$ and g. In other words isomorphic configurations have isomorphic features, and moreover the isomorphism is chosen in a uniform way.*

Example 2. Consider now the action of the group R_n of rigid motions on $\mathbb{R}^n$. We say that an *oriented configuration* in $\mathbb{R}^n$ is a finite multiset C together with a coloring with $n+1$ colors in style $(1, 1, \dots, 1)$, picking out a sequence of $n+1$ points $(v_0, v_1, \dots, v_n)$ of S in such a way that $\{v_0, v_1, \dots, v_n\}$ is affinely independent. To an oriented configuration $\langle C, v_0, v_1, \dots, v_n\rangle$ we associate an *orientation* $\phi(\langle C, v_0, v_1, \dots, v_n\rangle) = (u_1, u_2, \dots, u_n)$, where $u_i \in S^{n-1}$ is the unit vector in direction $v_i - v_0$. Note that for $g \in R$

$$\phi(g.\langle C, v_0, v_1, \dots, v_n\rangle) = \pi(g).\phi(\langle C, v_0, v_1, \dots, v_n\rangle),$$

where π is the canonical quotient map from R onto SO_n. Thus ϕ is an SO_n-feature of oriented configurations.

Definition 11. *Let $\mathcal{C}$ be a set of isomorphism classes of a-colored configurations. The* $\mathcal{C}$-list *of S is*

$$L_{\mathcal{C}}(S) = \{\langle C, c\rangle : C \in \mathcal{P}(S),\ c \text{ an } a\text{-colouring of } C,\ [\langle C, c\rangle]_G \in \mathcal{C}\}.$$

If f is an H-feature of such configurations then the $\mathcal{C}$-feature set of S *is the multiset*

$$F_{f,\mathcal{C}}(S) = \{f(\langle C, c\rangle) : \langle C, c\rangle \in L_{\mathcal{C}}(S)\}.$$

Remark 2. Note that the $\mathcal{C}$-list of S, and hence the $\mathcal{C}$-feature set F of S are *not* isomorphism invariants, so there is no hope that we will literally be able to reconstruct them. What we hope is that the isomorphism class $[F]_H$ of the feature set will be reconstructible.

Where it is unambiguous we shall suppress the qualifiers in H-feature, a-colored configuration, and $\mathcal{C}$-feature set.

Theorem 1. (Feature Theorem) *Let f be a feature of colored configurations (with associated homomorphism ϕ), $\mathcal{C}$ a set of isomorphism classes of colored configurations, each of size at most m, and S a multiset in X. Set $F = F_{f,\mathcal{C}}(S)$, the feature set of S. Then the k-deck of $H \rightarrowtail F$ is reconstructible from the mk-deck of $G \rightarrowtail S$. In particular if multisets in Y are reconstructible from their k-decks then $[F]_H$ is reconstructible from the mk-deck of S.*

Proof. Note first that there is a natural bijection between the feature set F and the $\mathcal{C}$-list $L = L_{\mathcal{C}}(S)$. Thus there is also a natural bijection between the r-submultisets of F and those of L. We will partition the r-submultisets $\{\langle C_i, c_i\rangle : i = 1, 2, \dots, r\}$ of L according to the set union (of multisets) $C = \bigcup_1^r C_i$: note that a given C may arise in many different ways. For a configuration C in X we say that a $\mathcal{C}$-*splitting* of C is a representation of C as a set union $C = \bigcup_1^r C_i$ together with a-colorings c_i for the C_i such that $[\langle C_i, c_i\rangle]_G \in \mathcal{C}$ for $i = 1, 2, \ldots, r$.We can then write

$$f(C) = \{\{f(\langle C_i, c_i\rangle)\}_1^r \,:\, \{\langle C_i, c_i\rangle\}_1^r \text{ is a } \mathcal{C}\text{-splitting of } C\}.$$

We obtain the multiset identity

$$\{K \subset F \,:\, |K| \leq k\} = \bigoplus_{\substack{C\in\mathcal{P}(S)\\ |C|\leq mk}} f(C),$$

and hence

$$\begin{aligned} D_k(H \rightarrowtail F) &= \{[K]_H \,:\, K \subset F, |K| \leq k\} \\ &= \bigoplus_{\substack{C\in\mathcal{P}(S\\ |C|\leq mk}} \{[L]_H \,:\, L \in f(C)\}. \end{aligned} \tag{1}$$

The final, crucial, observation is that the multiset of isomorphism classes

$$\{[L]_H \,:\, L \in f(C)\}$$

is reconstructible from $[C]_G$. To see this note that if $D \simeq C$, with say $g.C = D$, then the $\mathcal{C}$-splittings of C are isomorphic to the $\mathcal{C}$-splittings of D: if $C = \bigcup_1^k C_i$ and c_i are appropriate colorings then we set $D_i = g.C_i$ with colorings $d_i(x) = c_i(g^{-1}.x)$ for all $x \in D_i$. The set of features arising from $\{\langle D_i, d_i\rangle\}_1^k$ is isomorphic to that arising from $\{\langle C_i, c_i\rangle\}_1^k$. We have

$$\begin{aligned} \{f(\langle D_i, d_i\rangle)\} &= \{f(g.\,\langle C_i, c_i\rangle)\} \\ &= \{\phi(g).f(\langle C_i, c_i\rangle)\} \\ &= \phi(g).\,\{f(\langle C_i, c_i\rangle)\}\,. \end{aligned}$$

Thus, by (1), $\{[K]_H \,:\, K \subset F, |K| \leq k\}$ depends only on the collection of all isomorphism classes of elements of $\mathcal{P}(S)$ of size at most mk, which is the mk-deck of $G \rightarrowtail S$.

4. The regular action

One extremely natural action to consider is the regular action of a group on itself. In this section we prove that under fairly weak hypotheses the reconstructibility of $G \rightarrowtail X$ is at most that of the regular action. This result was proved for set reconstructibility in [25]; here we prove it for multiset reconstructibility, under weaker hypotheses. The following definition captures the restrictions we impose on our action.

Definition 12. *Let G be a group acting on a set X with orbits $\{X_\alpha\}_{\alpha\in I}$ for some index set I. We say that the action is* locally finite *if for every $\alpha \in I$ either $|X_\alpha| = 1$ or for one (hence every) $x \in X_\alpha$ the stabilizer of x is finite.*

Remark 3. Note that if G is finite then the action is certainly locally finite. Also the regular action of G on G is locally finite since all point stabilizers have size 1. The action of the group R of rigid motions of the plane on $\mathbb{R}^2$ is *not* locally finite since, for instance, uncountably many rotations fix the origin. However the action of R on the collection of all finite multisets in the plane with support of size at least 2 *is* locally finite.

Lemma 4. *If* $G \rightarrowtail X$ *is a transitive locally finite action then*

$$r_{\mathbb{N}}(G \rightarrowtail X) \leq r_{\mathbb{N}}(G).$$

Proof. Set $k = r_{\mathbb{N}}(G)$, and let S be a finite multiset in X. If $|X| = 1$ then we can reconstruct the number of points in S from the 1-deck of S, and *a fortiori* from the k-deck. Thus we may focus on the case in which every point in X has a finite stabilizer. We arrange to "pull back" S into G. Fix (arbitrarily) a point $x_0 \in X$. Define a multiset $\tilde{S}$ in G by

$$m_{\tilde{S}}(g) = m_S(g.x_0), \text{for all } g \in G.$$

$\tilde{S}$ is finite since S is finite and the action is locally finite. We will show that we can reconstruct the k-deck of $\tilde{S}$ from the k-deck of S, going via the functions $d_{S,k}$ and $d_{\tilde{S}}$ (discussed in Section 2). Let $\tilde{K}$ be a multiset in G of size at most k, and set $K = \tilde{K}.x_0$. [Note that K is a multiset in X of the same size as $\tilde{K}$, and that $m_K(x) = \sum_{g:g.x_0=x} m_{\tilde{K}}(g)$.] We have

$$\begin{aligned} d_{\tilde{S}}(\tilde{K}) &= \sum_{g\in G} \prod_{h\in\tilde{K}} m_{\tilde{S}}(gh) \\ &= \sum_{g\in G} \prod_{h\in\tilde{K}} m_S(gh.x_0) \\ &= d_{S,k}(K) \end{aligned}$$

By Lemma 2 we can reconstruct from the k-deck of S the values of $d_{\tilde{S}}$ on multisets of size at most k, and thence the k-deck of $\tilde{S}$. By hypothesis $\tilde{S}$ is reconstructible from its k-deck, i.e., we can determine $[\tilde{S}]_G$. Now we can determine S, up to the action of G, by picking $\tilde{T} \in [\tilde{S}]_G$ and letting T be the multiset in X defined by $m_T(x) = \left(\sum_{g:g.x_0=x} m_{\tilde{T}}(g)\right) / |\mathrm{Stab}(x_0)|$. We have $\tilde{T} \simeq \tilde{S}$, hence $|\mathrm{Stab}(x_0)|T \simeq \tilde{S}.x_0 = |\mathrm{Stab}(x_0)|S$. Therefore $T \simeq S$.

Before we prove the more general result covering non-transitive actions we prove a theorem which is useful in this context and also later. It addresses the issue of the reconstructibility of the direct sum of several copies of the same action.

Definition 13. *Given group actions* $G \rightarrowtail X$ *and* $G \rightarrowtail Y$ *their* disjoint union *is the (unique) action of* G *on the disjoint union of* X *and* Y *which restricts to the given actions. We write* $G \rightarrowtail \coprod^n X$ *for the disjoint union of* n *copies of* $G \rightarrowtail X$*. We call the individual terms in this disjoint union the* components *of* $\coprod^n X$*. Notice that this disjoint sum is* G*-isomorphic to the* n-colored version of X*, defined on* $X \times \{1, 2, \ldots, n\}$ *by* $g.(x, i) = (g.x, i)$.

Remark 4. Notice that the disjoint union of any collection of locally finite G-actions is locally finite; the stabilizer of a point x in a given component of the form $G \rightarrowtail X$ is simply the stabilizer of x in $G \rightarrowtail X$.

Theorem 2. *For all* $n \geq 1$

$$r_{\mathbb{N}}(G \rightarrowtail \textstyle\coprod^n X) = r_{\mathbb{N}}(G \rightarrowtail X).$$

Proof. The inequality $r_{\mathbb{N}}(G \rightarrowtail \coprod^n X) \geq r_{\mathbb{N}}(G \rightarrowtail X)$ is immediate. We show the reverse inequality as follows. Let $k = r_{\mathbb{N}}(G \rightarrowtail X)$, and suppose that S is a finite multiset in $\coprod^n X$. Let S_i be the portion of S contained in the i^{th} component of X. Since we can, rather easily, reconstruct the k-decks of the individual S_i from the k-deck of S simply by ignoring any equivalence classes $[K]_G$ for which K meets some component other than the i^{th}, it is clear that we can reconstruct the sequence of equivalence classes $([S_i])_1^n$. What we *need* to do is reconstruct the equivalence class $[(S_i)]$.

We claim that for any sequence of integers $(\lambda_i)_1^n$ we can reconstruct, from the k-deck of S, the k-deck of the multiset union

$$S(\lambda) = \bigoplus_1^n \lambda_i S_i,$$

regarded as a multiset in X. Note first that if $L_1, L_2, \ldots, L_n$ are multisets in X then, writing $(L_1, L_2, \ldots, L_n)$ for the multiset in $\coprod^n X$ meeting the i^{th} component in L_i, we have

$$d_{S,k}\left(L_1, L_2, \ldots, L_n\right) = \sum_{g \in G} \prod_{i=1}^n \prod_{x \in L_i} m_S(g.x)$$

Now, given a multiset K in G, let us compute $d_{S(\lambda)}(K)$.

$$\begin{aligned} d_{S(\lambda)}(K) &= \sum_{g \in G} \prod_{x \in K} m_{S(\lambda)}(g.x) \\ &= \sum_{g \in G} \prod_{h \in K} \left(\lambda_1 m_{S_1}(g.x) + \lambda_2 m_{S_2}(g.x) + \cdots + \lambda_n m_{S_n}(g.x)\right) \end{aligned}$$

Let us index the elements of K, so $K = \{x_1, x_2, \ldots, x_k\}$. When we expand out the product in the expression above for $d_{S(\lambda)}(K)$ we get one term for each way in which we can choose terms from the innermost sums for each x_i. Thus they are indexed by partitions $P_1, P_2, \ldots, P_k$ of $\{1, 2, \ldots, n\}$. For such a partition, define a corresponding splitting of K by $L_i = \{x_j : j \in P_i\}$. Note that $K = \bigoplus_1^n L_i$. Resuming our calculation we have

$$\begin{aligned} d_{S(\lambda)}(K) &= \sum_{g \in G} \sum_{P_1, P_2, \ldots, P_n} \prod_{i=1}^n \prod_{x \in L_i} \lambda_i m_{S_i}(g.x) \\ &= \sum_{P_1, P_2, \ldots, P_n} \prod_{i=1}^n \lambda_i^{|P_i|} \sum_{g \in G} \prod_{i=1}^n \prod_{x \in L_i} m_{S_i}(g.x) \\ &= \sum_{P_1, P_2, \ldots, P_n} \prod_{i=1}^n \lambda_i^{|P_i|} d_{S,k}\left(L_1, L_2, \ldots, L_n\right). \end{aligned}$$

Having expressed $d_{S(\lambda)}(K)$ in terms of $d_{S,k}$ we see that the k-deck of $S(\lambda)$, and hence $S(\lambda)$, is reconstructible from the k-deck of S.

To finish the proof, note that if we pick $\lambda_i = N^{i-1}$ for N sufficiently large (in particular $N > |S|$ will do) we can read off the individual S_i from the digits of the N-ary expansion of the multiplicities in $S(\lambda)$.

Theorem 3. *If $G \rightarrowtail X$ is a locally finite action then*

$$r_{\mathbb{N}}(G \rightarrowtail X) \leq r_{\mathbb{N}}(G).$$

Proof. We may suppose that X has only finitely many orbits, since any finite S intersects only finitely many orbits. Suppose then that the orbits in X are $X_1, X_2, \ldots, X_n$. Let $S_i = S \cap X_i$ for $i = 1, 2, \ldots, n$. We will use an approach similar to that in Lemma 4.

Pick (arbitrarily) $x_i \in X_i$, and define $\tilde{S}_i$ by

$$m_{\tilde{S}_i}(g) = m_{S_i}(g.x_0), \text{for all } g \in G.$$

$\tilde{S}_i$ is finite since S_i is finite and the action is locally finite. Clearly if we can reconstruct the equivalence class of $(\tilde{S}_i)_1^n \subset \coprod^n G$ we are done, since this determines (the equivalence class of) S as in Lemma 4. Now, by Theorem 2, $r_{\mathbb{N}}(G \rightarrowtail \coprod^n G) = r_{\mathbb{N}}(G)$, so if we let $k = r_{\mathbb{N}}(G)$ it is enough to show that we can reconstruct the k deck of $(\tilde{S}_i)_1^n \subset \coprod^n G$. If $\tilde{L}_i$ are multisets in G and we set $L_i = \tilde{L}_i.x_i$ we have

$$\begin{aligned}
d_{(\tilde{S}_i),k}((\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_n)) &= \sum_{g \in G} \prod_{i=1}^{n} \prod_{h \in \tilde{L}_i} m_{\tilde{S}_i}(gh) \\
&= \sum_{g \in G} \prod_{i=1}^{n} \prod_{h \in \tilde{L}_i} m_{S_i}(gh.x_i) \\
&= \sum_{g \in G} \prod_{i=1}^{n} \prod_{x \in L_i} m_{S_i}(g.x) \\
&= d_S(L_1 \cup L_2 \cup \cdots \cup L_n).
\end{aligned}$$

Thus the proof is complete.

Remark 5. Note that the hypothesis of local finiteness is necessary: for instance, it is easy to show that $r_{\mathbb{N}}(\mathbb{Z}) = 3$, while Pebody proved in [22] that $r_{\mathbb{N}}(\mathbb{Z} \rightarrowtail \mathbb{Z}_n)$ can be as high as 6.

Remark 6. If we consider the action of the symmetric group Σ_n on the set of pairs from $\{1, 2, \ldots, n\}$ then the problem becomes that of multigraph edge reconstruction. Theorem 3 shows that the reconstructibility of this action is at most $r_{\mathbb{N}}(\Sigma_n)$.

5. Subgroups

In this section we consider the situation in which a group G acting on X has a normal subgroup $N \triangleleft G$. We prove a bound relating the reconstructibility of $G \rightarrowtail X$ to the reconstructibility of $N \rightarrowtail X$ and $G/N \rightarrowtail \mathcal{M}(X)$. We would like to reconstruct $D_k(N \rightarrowtail S)$ from some deck of S. Of course as stated this is impossible, since that deck is not even an isomorphism invariant of S. Instead we will aim to reconstruct $[D_k(N \rightarrowtail S)]_G$ where the G-action concerned is that on multisets of multisets in X.

Our approach will be to treat the deck of $N \rightarrowtail S$ as the feature set of S where the relevant feature of a finite configuration $C \subset S$ is its N-isomorphism class $[C]_N$. We note that there is a natural action of $H = G/N$ on such isomorphism classes given by

$$(Ng).[K]_N = [g.K]_N.$$

This is clearly well defined since if $Ng = Ng'$ then $g = ng'$ for some $n \in N$ and therefore $g.C = n.(g'.C)$ and $[g.C]_N = [g'.C]_N$.

Theorem 4. *Suppose that $G \rightarrowtail X$ and $N \lhd G$ is a normal subgroup of G. Let $H = G/N$, and let Y be the set of all N-isomorphism classes of multisets in X. If the action of H on Y is locally finite then*

$$r_{\mathbb{N}}(G \rightarrowtail X) \leq r_{\mathbb{N}}(N \rightarrowtail X) r_{\mathbb{N}}(G/N).$$

Moreover the result holds "pointwise", in that if S is a multiset in X which is reconstructible up to the action of N from the k-deck of $N \rightarrowtail S$ then it is reconstructible up to the action of G from the k'-deck of $G \rightarrowtail S$, where $k' = kr_{\mathbb{N}}(G/N)$.

Proof. Set $m = r_{\mathbb{N}}(N \rightarrowtail X)$ and let S be a finite multiset in X. Let $\mathcal{C}$ be the set of all G-isomorphism classes of subsets of S of size at most m. Let $H = G/N$. We will define an H-feature of configurations in X by $f(C) = [C]_N$. Letting π be the canonical projection from G onto H we have

$$f(g.C) = [g.C]_N = \pi(g).[C]_n = \pi(g).f(C).$$

Thus f is indeed an H-feature with associated homomorphism π. Theorem 3 implies that $r_{\mathbb{N}}(H \rightarrowtail Y) \leq r_{\mathbb{N}}(H)$. Thus by Theorem 1 we can reconstruct the H-isomorphism class of the feature set

$$F = F_{f,\mathcal{C}}(S) = \{[C]_N : C \subset S, [C]_G \in \mathcal{C}\}$$

from the $(mr_{\mathbb{N}}(H))$-deck of S. But F is simply the k-deck of $N \rightarrowtail S$, and $\pi(g).F$ is similarly the k-deck of $N \rightarrowtail g.S$. Thus any representative we pick for $[F]_H$ is the k-deck of $g.S$ for some $g \in G$. By the choice of k we can reconstruct from this deck the isomorphism class $[g.S]_N$, but this clearly allows us to reconstruct $[S]_G$.

A simple special case of Theorem 4 proves a variant of a conjecture from [23]. Let use define the *set reconstructibility* of a group action $G \rightarrowtail X$ to be the smallest k such that all subsets of X are reconstructible from their k-decks. We write $r(G \rightarrowtail X)$ for the set reconstructibility, and $r(G)$ for the set reconstructibility of the (left) regular action of G. In [23] the authors conjectured that for all finite groups G and H

$$r(G \times H) \leq r(G)r(H).$$

We will show that the conjecture is true with $r_{\mathbb{N}}$ in place of r. We prove a simple lemma first.

Lemma 5. *If $G \rightarrowtail X$ is locally finite then so is the action of G on finite multisets in X. Moreover if Y is an arbitrary disjoint union of copies of X then the action of G on finite multisets in Y is locally finite.*

Proof. Let S be a finite multiset in X. Letting S' be the sub-multiset of S consisting of those x with finite stabilizer we either have $S' = \emptyset$ in which case $|\mathrm{Orb}(S)| = 1$, or $S' \neq \emptyset$. In this case we know that any $g \in \mathrm{Stab}(S)$ induces a permutation of S'. Given a permutation π of S' there are only finitely many group elements g such that $g.x = \pi(x)$ for all $x \in S'$. Since there are only finitely many permutations of S' we have that $\mathrm{Stab}(S)$ is finite. The second part follows since the action of G on Y is locally finite by Remark 4.

Theorem 5. *If G and H are groups then*

$$r_{\mathbb{N}}(G \times H) \leq r_{\mathbb{N}}(G) r_{\mathbb{N}}(H).$$

Proof. Clearly G is a normal subgroup of $G \times H$ with quotient H. Note that $G \rightarrowtail G \times H$ is simply the disjoint union of $|H|$ copies of $G \rightarrowtail G$ and hence

$$r_{\mathbb{N}}(G \rightarrowtail G \times H) \leq r_{\mathbb{N}}(G).$$

The action of H on finite multisets in $G \times H$ is locally finite by Lemma 5 and $r_{\mathbb{N}}(H \rightarrowtail \mathcal{M}(G \times H)) \leq r_{\mathbb{N}}(H)$ by Theorem 3. Thus, by Theorem 4,

$$r_{\mathbb{N}}(G \times H) \leq r_{\mathbb{N}}(G) r_{\mathbb{N}}(H).$$

The previous theorem shows that $r_{\mathbb{N}}(G^n) \leq (r_{\mathbb{N}}(G))^n$, but this is unlikely to be the correct rate of growth. For instance, it is well known that $r(G) \leq \log_2 |G|$. Thus, for finite groups G, $r(G^n)$ grows at most linearly with n. This seems likely to be the true behavior of $r_{\mathbb{N}}(G^n)$ also. More precisely we believe the following.

Conjecture 1. If G is a group with finite (multiset) reconstructibility then

$$r_\infty = \lim_{n\to\infty} \frac{r_{\mathbb{N}}(G^n)}{n}$$

exists and is finite.

Conjecture 1 holds for finite Abelian groups by results of Pebody [22].

6. Applications

In this section we give some applications of the methods developed in the previous sections. Firstly we show that Theorem 4 allows us to give an alternative proof of the following result from [21].

Corollary 1. *Finite multisets of $\mathbb{R}^2$ are reconstructible, up to the action of the group R of rigid motions, from their 18-decks.*

Proof. Consider the subgroup $T \lhd R$ consisting of the translations. The quotient R/T is isomorphic to the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In [22] Pebody proved that $r_{\mathbb{N}}(\mathbb{T}) = 6$, and it is straightforward to check (see, e.g., [21] or [24]) that $r(T \rightarrowtail \mathbb{R}^2) = 3$. The action of R/T on T-isomorphism classes of multisets in $\mathbb{R}^2$ is locally finite since a multiset K in the plane has either support of size 1, in which case the orbit $\mathrm{Orb}(H \rightarrowtail [K]_T)$ has size 1, or else K has only finitely many rotational symmetries. Now we can apply Theorem 4.

Similarly we can show that finite multisets in the plane are reconstructible up to isometry from their 36-decks.

Corollary 2. *Let I be the group of isometries of $\mathbb{R}^2$. Then*

$$r(I \rightarrowtail \mathbb{R}^2) \leq 2r(R \rightarrowtail \mathbb{R}^2).$$

In particular $r(I \rightarrowtail \mathbb{R}^2) \leq 36$. Similarly, if I_n is the group of isometries of $\mathbb{R}^n$ then

$$r(I_n \rightarrowtail \mathbb{R}^n) \leq 2r(R_n \rightarrowtail \mathbb{R}^n).$$

Proof. The group of rigid motions is of index 2, and hence normal, in I. Since I/R is finite, its action on R-isomorphism classes of multisets in $\mathbb{R}^2$ is certainly locally finite. By Theorem 4 and Corollary 1,

$$r_{\mathbb{N}}(I \rightarrowtail \mathbb{R}^2) \leq r_{\mathbb{N}}(R \rightarrowtail \mathbb{R}^2) r_{\mathbb{N}}(\mathbb{Z}_2) \leq 36,$$

since clearly $r_{\mathbb{N}}(\mathbb{Z}_2) = 2$. The argument is the same in n dimensions.

Remark 7. We give now another, quite similar but instructive example. Let H be the group of homotheties of the plane: maps composed of a rigid motion followed by a map of the form $x \mapsto \lambda x$ for some $\lambda \in \mathbb{R}_+$. The group of rigid motions R is normal in H and the quotient is isomorphic to $(\mathbb{R}, +)$. It is easy to see that $r_{\mathbb{N}}(\mathbb{R}) = 3$, and moreover that the action of $\mathbb{R}$ on R-isomorphism classes of finite multisets in $\mathbb{R}^2$ is locally finite (indeed all isomorphism classes of multisets of support greater than 1 have trivial $\mathbb{R}$-stabilizer). Hence

$$r_{\mathbb{N}}(H \rightarrowtail \mathbb{R}^2) \leq r_{\mathbb{N}}(R \rightarrowtail \mathbb{R}^2) r_{\mathbb{N}}(\mathbb{R}) \leq 54.$$

This bound is however quite far from the correct answer, as we shall now show.

We use the following lemma. Note that the proof runs along similar lines to that of the Feature Theorem.

Lemma 6. *Let $G \rightarrowtail X$ be a group action, and suppose $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t$ are collections of isomorphism classes of configurations in X such that $|C_i| \leq k_i$ for all $C_i \in \mathcal{C}_i$. If S is a finite multiset in X then it is possible to reconstruct the multiset*

$$\mathcal{S} = \{[(S_1, S_2, \ldots, S_t)]_G \,:\, S_i \subset S, [S_i]_G \in \mathcal{C}_i\}$$

knowing only the multiplicities in the $(m_1 + m_2 + \cdots + m_t)$-deck of S of multisets of the form $S_1 \cup S_2 \cup \cdots \cup S_t$ with $[S_i] \in \mathcal{C}_i$. Moreover, from $\mathcal{S}$ we can immediately deduce, for any i, the multiplicities in the m_i-deck of multisets S_i with $[S_i] \in \mathcal{C}_i$.

Proof. For a configuration C in X we say that a $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t)$-*splitting* of C is a representation of C as a set union $C = \bigcup_1^t S_i$ such that $[C_i]_G \in \mathcal{C}_i$ for $i = 1, 2, \ldots, t$. The multiset

$$\{[(S_1, S_2, \ldots, S_t)]_G \,:\, S_1, S_2, \ldots, S_t \text{ is a } (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t)\text{-splitting of } C\}.$$

can clearly be determined from the equivalence class $[C]_G$. Noting that if C has a $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t)$-splitting it has size at most $m_1 + m_2 + \cdots + m_t$, and that

$$\mathcal{S} = \bigoplus_{C \subset S} \{[(S_1, S_2, \ldots, S_t)]_G \,:\, S_1, S_2, \ldots, S_t \text{ is a } (\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_t)\text{-splitting of } C\}$$

it is clear that $\mathcal{S}$ is reconstructible from the $m_1 + m_2 + \cdots + m_t$ deck of S. The last statement of the lemma is straightforward.

Theorem 6. $r_{\mathbb{N}}(H \rightarrowtail \mathbb{R}^2) \leq r_{\mathbb{N}}(R \rightarrowtail \mathbb{R}^2) + 4.$

Proof. Suppose that S is a finite multiset in $\mathbb{R}^2$. From the 4-deck of $H \rightarrowtail S$ we can recover the value of

$$\Lambda(S) = \max\{|x_1 - x_2|/|x_3 - x_4| \,:\, x_1, x_2, x_3, x_4 \in S, x_3 \neq x_4\}.$$

Set $k = r_{\mathbb{Q}}(R \rightarrowtail \mathbb{R}^2)$. Letting $\mathcal{C}_1$ be the set of all H-isomorphism classes of multisets of size at most k in $\mathbb{R}^2$ and $\mathcal{C}_2$ be the set of all H-isomorphism classes of subsets $C \subset S$ of size at most 4 with $\Lambda(C) = \Lambda(S)$. By Lemma 6 we can reconstruct

$$\mathcal{S} = \{[(S_1, S_2)]_H \,:\, S_1, S_2 \subset S, [S_1]_H \in \mathcal{C}_1, [S_2]_H \in \mathcal{C}_2\}$$

from the $(k+4)$-deck of $H \rightarrowtail S$. Let S_0 be an isomorphic copy of S scaled so that the shortest distance between points is 1. We obtain the k-deck of $R \rightarrowtail S_0$ as follows: for each $[(S_1, S_2)]_H \in \mathcal{S}$ scale the pair (S_1, S_2) so that the shortest distance between points in S_2 is 1 and then take S_1. (Of course the multiplicity of each deck element is too large by a factor of $|\{C \subset S \,:\, [C]_H \in \mathcal{C}_2\}|$.) By hypothesis $[S_0]_R$ is k-reconstructible.

Our last example is that of the group R_n of rigid motions acting on $\mathbb{R}^n$. In [21] the present authors raised the question of whether $r_{\mathbb{N}}(R_n \rightarrowtail \mathbb{R}^n)$ is finite for all n. We sketch a proof here that this problem reduces, as it did in the case of the plane, to the question of whether the reconstructibility of SO_n acting on S^{n-1} is finite.

We say that a multiset $S \subset \mathbb{R}^n$ has *full dimension* if the affine span of S is the whole of $\mathbb{R}^n$.

Theorem 7. *If $n \geq 3$ and $k = r_{\mathbb{N}}(SO_n \rightarrowtail S^{n-1})$ is finite then there exists a constant k' such that all finite multisets in $R_n \rightarrowtail \mathbb{R}^n$ of full dimension are k'-reconstructible.*

Proof. Suppose that S is a finite multiset in $\mathbb{R}^n$ of full dimension. Let us set $R = R_n$, and also abbreviate the subgroup of translations to T. As in Corollary 1 it is enough to reconstruct the 3-deck of $T \rightarrowtail S$, or to be more precise, the R-equivalence class of that deck. The issue of local finiteness arises, since for $n \geq 4$ all multisets of size 3 have T-equivalence classes with infinite stabilizer in SO_n. We address this problem by using Lemma 6 again. Let $\mathcal{F}$ be the set of all T-isomorphism classes of multisets of size $n+1$ with full dimension, and let $\mathcal{T}$ be the set of all T-isomorphism classes of multisets of size 3. By Lemma 6, it is sufficient to reconstruct that portion of the $(n+4)$-deck of $T \rightarrowtail S$ consisting of sets of the form $A \cup B$ with $[A]_T \in \mathcal{T}$ and $[B]_T \in \mathcal{F}$: this would give us the 3-deck of $T \rightarrowtail S$, from which we can reconstruct S.

Define the multiset

$$\mathcal{B} = \{[A \cup B]_T \,:\, A, B \subset S, [A]_T \in \mathcal{T}, [B]_T \in \mathcal{F}\}.$$

We aim to reconstruct $[\mathcal{B}]_R$: note that $[\mathcal{B}]_R = [B]_{SO_n}$, since $\mathcal{B}$ consists of T-equivalence classes. Our strategy will be to associate translation-invariant features to sub-multisets of S. These features will allow us to orient the various equivalence classes $[C]_T \in \mathcal{B}$ with respect to one another.

Recall that in Example 2 we associated with any oriented multiset an "orientation"

$$\phi(\langle C, v_0, v_1, \ldots, v_n\rangle) \in (S^{n-1})^n.$$

From this orientation we construct a feature which is a subset of S^{n-1}. For distinct $u_1, u_2, \ldots, u_n$ in S^{n-1} and $\delta > 0$ we define $F_\delta(u_1, u_2, \ldots, u_n)$ to be the colored set consisting of u_i with color i for $1 \le i \le n$ and for each $1 \le i < j \le n$, two points of color 0 positioned along the shorter arc of the great circle from u_i to u_j: one point of color 0 is placed fraction δ of the way from u_i to u_j, the other is placed fraction $\delta + \delta^2$ of the way. (Note that this is a set if δ is sufficiently small; note also that F_δ is only well defined provided no pair of the u_i is antipodal.)

Now let Y be the direct sum of many copies of $SO_n \rightarrowtail S^{n-1}$, one for each R-equivalence class of a multiset of size at most $n+4$. The feature that we associate with an oriented subset $\langle C, v_0, v_1, \ldots, v_n\rangle$ is the subset of Y consisting of $F_\delta(\phi(\langle C, v_0, v_1, \ldots, v_n\rangle))$ in the component corresponding to $[B]_R$. Let F be the multiset of features arising from all allowed configurations in S. The Feature theorem then states that the k-deck of $SO_n \rightarrowtail F$ can be reconstructed from the $(n+4)k$-deck of $R \rightarrowtail S$. Since, by Theorem 2, neither direct sums nor coloring make reconstruction any harder, this means we can reconstruct $[F]_{SO_n}$ from the same deck. Pick a representative $F' \in [F]_{SO_n}$. In the $[C]_R$ component of F' we can determine, for δ sufficiently small, which points labelled 1, say, go with which points labelled 2, etc. Thus from F' we can in fact determine all the orientations $\phi(C')$ of subsets C' R-isomorphic to C. By our hypothesis concerning ϕ this in turn determines $[C']_T$ for all such subsets. Thus we have reconstructed the relevant portion of the $(n+4)$-deck of $T \rightarrowtail S'$, where S' is some multiset R-isomorphic to S. By our earlier comment concerning Lemma 6 we are done.

Corollary 3. *If $r_{\mathbb{N}}(SO_n \rightarrowtail S^{n-1})$ is finite then so is $r_{\mathbb{N}}(R_m \rightarrowtail \mathbb{R}^m)$ for all $m \le n$.*

Proof. We will show first that all full dimensional finite multisets in $\mathbb{R}^m$ for $m \le n$ are uniformly finitely reconstructible. By Theorem 7 we know that there is some k' such that all full dimension finite multisets in $\mathbb{R}^n$ are reconstructible from their k'-decks. We will show that all full dimensional multisets in $\mathbb{R}^m$ are k'-reconstructible, for all $m \le n$. Suppose S is a multiset in $\mathbb{R}^m$ of full dimension, where $m < n$. Choose $D \in \mathbb{R}$ with D larger than twice the largest distance appearing in S. Now the k'-deck of $S \times \{0, 1, D\}$ is straightforward to determine from the k'-deck of S, and hence, by downwards induction on m, $S \times \{0, 1, D\}$ is reconstructible up to a rigid motion in $\mathbb{R}^{m+1}$. This is clearly sufficient to reconstruct S up to a rigid motion of $\mathbb{R}^m$.

Now suppose that S is a finite multiset not of full dimension in $\mathbb{R}^m$, $m \le n$. We can tell the dimension d of S from (at worst) its $(n+1)$-deck. Without loss of generality we may assume that $S \subset \mathbb{R}^d$, and the $2k'$-deck of $R_m \rightarrowtail S$ can be interpreted as the $2k'$-deck of $I_d \rightarrowtail S$. By the "pointwise" version of Theorem 4 and Corollary 2 we know that from this we can reconstruct the I_d isomorphism class of S. This in turn determines $[S]_{R_m \rightarrowtail \mathbb{R}^m}$.

The first open case in this area is the question of whether $r_{\mathbb{N}}(R_3 \rightarrowtail \mathbb{R}^3) < \infty$. By the result above this reduces to the question of whether the reconstructibility of $SO_3 \rightarrowtail S^2$ is finite.

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Received:
Final version received: