Basic existence and uniqueness results for positive solutions to nonlinear dynamic equations on time scales

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Abstract

This paper focuses on the qualitative and quantitative properties of solutions to certain nonlinear dynamic equations on time scales. We present some new sufficient conditions under which these general equations admit a unique, positive solution. These positive (and hence non–oscillatory) solutions: extend across unbounded intervals; and tend to a finite limit as the independent variable becomes large and positive. Our methods include: the Banach fixed–point theorem, including the method of Picard iterations; and weighted norms and metrics in the time scale setting. Due to the wide–ranging nature of dynamic equations on time scales our results are novel: for ordinary differential equations; for difference equations; for combinations of the two areas; and for general time scales.

Keywords and Phrases: positive solution; Second order dynamic equations; terminal condition; time scale.

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1 Introduction

"Oscillation theory" forms an important area in the qualitative analysis of differential equations. This includes the existence of oscillatory or nonoscillatory solutions to second–order, ordinary differential equations. A solution to a (real) second–order ordinary differential equation is termed to be "oscillatory on $[a,\infty)$" provided that it exists on $[a,\infty)$ and has an arbitrarily large number of zeros on $[a,\infty)$. A nontrivial solution that has a finite number of zeros is called nonoscillatory.

The theory of oscillations has a long and rich history dating back to the work of Sturm in 1836 [19, 14] and has since enjoyed numerous applications to a wide range of areas in science, engineering and technology [17, pp xiii–xiv]. Moreover, interest in the area continues to grow as oscillation theory naturally stimulates the emergence of new mathematical theories and methods [1, p.xiv], including the improvement of modern numerical methods and their application to computing.

One exciting aspect of these new movements includes linking the field of oscillation theory with solutions to "dynamic equations on time scales". The area of dynamic equations on time scales is a new, modern and progressive component of applied analysis that acts as the framework to effectively describe processes that feature both continuous and discrete elements. Created by Hilger in 1990 [13] and developed by others (see [3, 5, 6, 15, 20] and references therein), this novel and fascinating type of mathematics is more general and versatile than the traditional theories of differential and difference equations as it can, under one framework, mathematically describe continuous–discrete hybrid processes and hence is the optimal way forward for accurate and malleable mathematical modelling. In fact, the field of dynamic equations on time scales contains, links and extends the classical theory of differential and difference equations.

In this paper, we are concerned with proving the existence of positive (and so nonoscillatory) solutions to the nonlinear dynamic equation

$$[r(t)x^{\Delta}(t)]^{\Delta} + F(t, x^{\sigma}(t), x^{\Delta\sigma}(t)) = 0, \quad t \in [a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}; \tag{1.1}$$

where $F : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ may be a nonlinear function; t is from a so–called "time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$); x^{Δ} is the generalised "delta" derivative of x; $a \in \mathbb{T}$; $r : [a,\infty)_{\mathbb{T}} \to (0,\infty)$; and $x^{\sigma} := x \circ \sigma$ with σ a function to be defined a little later.

We will also consider the following special case of (1.1)

$$[r(t)x^{\Delta}(t)]^{\Delta} + F(t, x^{\sigma}(t)) = 0, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{1.2}$$

If $\mathbb{T} = \mathbb{R}$ then $x^{\Delta} = x'$ with $\sigma(t) = t$ and (1.1) becomes the familiar ordinary differential equation

$$[r(t)x'(t)]' + F(t, x(t), x'(t)) = 0, \quad t \in [a,\infty);$$

while if $\mathbb{T} = \mathbb{Z}$ then $x^{\Delta} = x(t+1) - x(t) := \Delta x(t)$ with $\sigma(t) = t+1$ and (1.1) becomes the well–known difference equation

$$\Delta[r(t)\Delta x(t)] + F(t, x(t+1), \Delta x(t+1)) = 0, \quad t \in \{a, a+1, \ldots\}.$$

There are many more time scales than just $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ and hence there are many more dynamic equations.

We shall assume that initial value problems for (1.1) *and* (1.2) *have unique solutions.*

We shall be primarily concerned with proving the existence of positive solutions which are asymptotic to a positive limit at ∞.

In the field of differential equations, some of the earliest work concerned with proving the existence of positive solutions which are asymptotic to a positive limit at ∞ dates back to the result of Atkinson [2] who showed that if $\int_0^\infty tf(t)dt < \infty$, then the superlinear equation

$$x'' + f(t)g(x) = 0 \tag{1.3}$$

has a bounded, nonoscillatory solution. This idea was extended in various ways by many authors over the last 50 years. Recently, Dubé and Mingarelli [9] presented a general nonoscillation result, which unifies a number of cases for the equation

$$x'' + f(t, x) = 0 \tag{1.4}$$

where

$$f(t, x) \geq 0, \quad \forall\ (t, x) \in [0, \infty) \times [0, \infty).$$

This result was further extended by Wahlén in [21] and by Ernström in [12], using a renormalisation technique.

Here we present three sorts of results. In the first result we show existence of a positive solution on a given fixed interval and with a given (asymptote) at ∞. In the second case, the limit is prescribed and the solution is shown to be eventually positive, and in the special case of equation (1.2), we fix an interval of positivity but not the limit.

These positive (and hence non–oscillatory) solutions: extend across unbounded intervals; and tend to a finite limit as the independent variable becomes large and positive. Our methods include: the Banach fixed–point theorem, including the method of Picard iterations; and weighted norms and metrics in the time scale setting. Due to the wide–ranging nature of dynamic equations on time scales our results are novel: for ordinary differential equations when $r \not\equiv 1$; for difference equations; for combinations of the two areas; and for general time scales. We present some examples to illustrate how our results advance existing knowledge.

For more recent results on oscillation and nonoscillation for solutions to dynamic equations on time scales we refer the reader to [7] and [11] and the references therein.

2 The time scales

To understand the notation used above, some preliminary definitions are needed, which are now presented. For more detail see [5, Chapter 1].

Definition 2.1 *A time scale* $\mathbb{T}$ *is a nonempty closed subset of the real numbers* $\mathbb{R}$*.*

Since a time scale may or may not be connected, the concept of the jump operator is useful to define the generalised derivative x^Δ of a function x.

Definition 2.2 *The forward (backward) jump operator* $\sigma(t)$ *at* t *for* $t < \sup \mathbb{T}$ *(respectively* $\rho(t)$ *at* t *for* $t > \inf \mathbb{T}$*) is given by*

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\},) \quad \textit{for all } t \in \mathbb{T}.$$

Define the graininess function $\mu : \mathbb{T} \to [0, \infty)$ *as* $\mu(t) := \sigma(t) - t$*.*

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition 2.3 *The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$, then the point t is called right–scattered; while if $\rho(t) < t$, then t is termed left–scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then the point t is called right–dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then we say t is left–dense.*

If $\mathbb{T}$ has a left–scattered maximum value m, then we define $\mathbb{T}^\kappa := \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^\kappa := \mathbb{T}$.

The following gives a formal $\varepsilon - \delta$ definition of the generalised delta derivative.

Definition 2.4 *Fix $t \in \mathbb{T}^\kappa$ and let $x : \mathbb{T} \to \mathbb{R}^n$. Define $x^\Delta(t)$ to be the vector (if it exists) with the property that given $\varepsilon > 0$ there is a neighbourhood U of t with*

$$|[x_i(\sigma(t)) - x_i(s)] - x_i^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call $x^\Delta(t)$ the delta derivative of $x(t)$ and say that x is delta–differentiable.

Converse to the delta derivative, we now state the definition of the delta integral.

Definition 2.5 *If $K^\Delta(t) = k(t)$ then define the (Cauchy) delta integral by*

$$\int_a^t k(s)\Delta s = K(t) - K(a).$$

If $\mathbb{T} = \mathbb{R}$ then $\int_a^t k(s)\Delta s = \int_a^t k(s)\ ds$, while if $\mathbb{T} = \mathbb{Z}$ then $\int_a^t k(s)\Delta s = \sum_a^{t-1} k(s)$. Once again, there are many more time scales than just $\mathbb{R}$ and $\mathbb{Z}$ and hence there are many more delta integrals. For a more general definition of the delta integral see [5].

The following theorem will be fundamental.

Theorem 2.6 *[13] Assume that $k : \mathbb{T} \to \mathbb{R}^n$ and let $t \in \mathbb{T}^\kappa$.*
(i) If k is delta–differentiable at t then k is continuous at t.
(ii) If k is continuous at t and t is right–scattered then k is delta–differentiable at t with

$$k^\Delta(t) = \frac{k(\sigma(t)) - k(t)}{\sigma(t) - t}.$$

(iii) If k is delta–differentiable and t is right–dense then

$$k^\Delta(t) = \lim_{s \to t} \frac{k(t) - k(s)}{t - s}.$$

(iv) If k is delta–differentiable at t then $k(\sigma(t)) = k(t) + \mu(t)k^\Delta(t)$.

For brevity, we will write x^σ to denote the composition $x \circ \sigma$.

The following gives a generalised idea of continuity on time scales.

Definition 2.7 *Assume $k : \mathbb{T} \to \mathbb{R}^n$. Define and denote $k \in C_{rd}(\mathbb{T})$ as right–dense continuous (rd–continuous) if: k is continuous at every right–dense point $t \in \mathbb{T}$; and $\lim_{s \to t^-} k(s)$ exists and is finite at every left–dense point $t \in \mathbb{T}$. For functions F of two variables, $t \in \mathbb{T}$ and $u \in R$, we say F is rd–continuous if it is rd–continuous in t and continuous in u.*

Of particular importance is the fact that every C_{rd} function is delta–integrable [5, Theorem 1.73].

3 Banach space construction

In this section we construct a suitable Banach space to accommodatete solutions to (1.1) by introducing a novel weighted norm in the time scale environment.

Since we will assume that r and F are rd–continuous functions in (1.1) and (1.2), we are searching for the existence and uniqueness of solutions $x : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ such that $x^{\Delta} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ is continuous and $x^{\Delta\Delta} : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ is rd–continuous.

We denote, by $C_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$, the space of continuously delta differentiable functions $x : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ that satisfy

$$\sup_{t\in[a,\infty)_{\mathbb{T}}} |x(t)| < \infty;$$

and couple this linear space with the norm

$$\|x\|_0^{\mathbb{T}} := \sup_{t\in[a,\infty)_{\mathbb{T}}} |x(t)|$$

so that we obtain a Banach space. In particular, the following closed subspace of $C_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$

$$\mathcal{H} := \{x \in C_{\mathbb{T}}([a,\infty)_{\mathbb{T}}) : \ x(t) \geq 0, \ t \in [a,\infty)_{\mathbb{T}}\}$$

coupled with the above sup–norm furnishes a Banach space.

Similarly, we denote, by $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$, the space of continuously delta differentiable functions $x : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ that satisfy

$$\max\left\{ \sup_{t\in[a,\infty)_{\mathbb{T}}} |x(t)|, \ \sup_{t\in[a,\infty)_{\mathbb{T}}} |x^{\Delta}(t)| \right\} < \infty$$

and couple this linear space with the norm

$$\|x\|_1^{\mathbb{T}} := \max\left\{ \sup_{t\in[a,\infty)_{\mathbb{T}}} |x(t)|, \ \sup_{t\in[a,\infty)_{\mathbb{T}}} |x^{\Delta}(t)| \right\} \tag{3.1}$$

to obtain a Banach space. For a non–negative constant M, we define the closed subspace of $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ by

$$\mathcal{X}_M := \{x \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}) : \ 0 \leq x(t) \leq M; \ x^{\Delta}(t) \geq 0; \ t \in [a,\infty)_{\mathbb{T}}\}; \tag{3.2}$$

and form a Banach space when coupling this with the norm in (3.1).

Finally, for any bounded function $w : [a,\infty)_{\mathbb{T}} \to [c,d]$ with $0 < c \leq d < \infty$, we introduce the weighted norm $\|\cdot\|_w$ on $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ by

$$\|x\|_w := \max\left\{ \sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{|x(t)|}{w(t)}, \ \sup_{t\in[a,\infty)_{\mathbb{T}}} \frac{|x^{\Delta}(t)|}{w(t)} \right\}.$$

We note that the existence of positive numbers c and d above ensure that $\|\cdot\|_w$ and $\|\cdot\|_1^{\mathbb{T}}$ are equivalent norms, as

$$\frac{\|x\|_1^{\mathbb{T}}}{d} \leq \|x\|_w \leq \frac{\|x\|_1^{\mathbb{T}}}{c}, \quad \forall x \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}).$$

Thus we observe that $(C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}), \|\cdot\|_w)$ is also a Banach space.

Weighted norms date back to the pioneering arguments of Bielecki [8, pp.25–26], [10, pp.153–155], [18, p.44] and have been recently introduced into the time scale setting by Tisdell and Zaidi [20].

Our particular weighted norm of interest will involve the exponential function on a time scale and we now provide a short discussion of this special function.

Define the so–called set of regressive functions, $\mathcal{R}$, by

$$\mathcal{R} := \{p \in C_{rd}(\mathbb{T};\mathbb{R}) \text{ and } 1 + p(t)\mu(t) \neq 0, \ \forall t \in \mathbb{T}\}$$

and the set of positively regressive functions, $\mathcal{R}^+$, by

$$\mathcal{R}^+ := \{p \in C_{rd}(\mathbb{T}) \text{ and } 1 + p(t)\mu(t) > 0, \ \forall t \in \mathbb{T}\}. \tag{3.3}$$

For $p \in \mathcal{R}$ we define [5, Theorem 2.35] the exponential function $e_p(\cdot, a)$ on the time scale $\mathbb{T}$ as the unique solution to the scalar initial value problem

$$x^{\Delta} = p(t)x, \quad x(t_0) = 1.$$

If $p \in \mathcal{R}^+$ then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$ [5, Theorem 2.48]. Finally, for $p \in \mathcal{R}$ we define $\ominus p$ by

$$(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)}, \quad t \in [a,\infty)_{\mathbb{T}}.$$

We now provide our first Banach space related result.

Lemma 3.1 *Let $p : [a,\infty)_{\mathbb{T}} \to (0,\infty)$ be a rd–continuous function such that*

$$\int_a^{\infty} p(t)\ \Delta t < \infty. \tag{3.4}$$

Then for $w(t) := e_{\ominus p}(t,a)$ the pair $(C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}), \|\cdot\|_w)$ forms a Banach space.

Proof Let

$$w(t) := e_{\ominus p}(t,a), \quad t \in [a,\infty)_T.$$

Since $p \in \mathcal{R}$, w is well defined and since $p > 0$ on $[a,\infty)_{\mathbb{T}}$ we have $p \in \mathcal{R}^+$. Thus, by [5, Theorem 2.44] $w > 0$ on $[a,\infty)_{\mathbb{T}}$. By [5, Theorem 2.71], w is the unique solution of the initial value problem

$$w^{\Delta}(t) = -p(t)w^{\sigma}(t), \quad w(a) = 1. \tag{3.5}$$

We show that $0 < L \leq w(t) \leq 1$ for $t \in [a,\infty)_{\mathbb{T}}$ where $L := \lim_{t\to\infty} w(t)$. Thus, the equivalence of the norms $\|\cdot\|_w$ and $\|\cdot\|_1^{\mathbb{T}}$ will be established and by the earlier discussion in this section the pair $(C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}), \|\cdot\|_w)$ will form a Banach space.

From (3.5) we see $w^{\Delta}(t) < 0$ for $t \in [a,\infty)_{\mathbb{T}}$ and so w is decreasing on $[a,\infty)_{\mathbb{T}}$ and thus $w \leq 1$ on $[a,\infty)_{\mathbb{T}}$.

We claim (3.4) implies that

$$L := \lim_{t\to\infty} w(t) > 0.$$

To see this note that

$$w(t) := e_{\ominus p}(t,a) = \exp\left(\int_a^t \xi_{\mu(s)}((\ominus p)(s))\ \Delta s\right), \quad t \in [a,\infty)_{\mathbb{T}},$$

where ξ_h is the cylinder transformation [5, Definition 2.21] defined by

$$\xi_h(x) := \begin{cases} \frac{1}{h} Log[1+hx], & h > 0; \\ x, & h = 0; \end{cases}$$

and Log is the principal logarithmic function.

There are two cases to discuss: $\mu(s) = 0$; and $\mu(s) > 0$.

If $\mu(s) = 0$, then $\xi((\ominus p)(s)) = -p(s)$; while if $\mu(s) > 0$ we have

$$\begin{aligned} \xi_{\mu(s)}((\ominus p)(s)) &= \frac{1}{\mu(s)} Log[1+\mu(s)(\ominus p)(s)] \\ &= \frac{1}{\mu(s)} Log[1+\mu(s)(\ominus p)(s)] \\ &= \frac{1}{\mu(s)} log\left[\frac{1}{1+\mu(s)p(s)}\right] \\ &= -\frac{1}{\mu(s)} log[1+\mu(s)p(s)] \\ &\geq -p(s) \end{aligned}$$

where we have used the inequality

$$\log(1+y) \leq y, \quad \forall\, y \geq 0.$$

Therefore, in either case we have

$$\xi_{\mu(s)}((\ominus p)(s)) \geq -p(s) \quad \text{for all} \quad s \in [a,\infty)_{\mathbb{T}}.$$

It follows from (4.4) that

$$L := \lim_{t\to\infty} w(t) > 0.$$

In particular, we have

$$0 < L \leq w(t) \leq 1, \quad t \in [a,\infty)_{\mathbb{T}}; \tag{3.6}$$

so that $\|\cdot\|_w$ and $\|\cdot\|_1^{\mathbb{T}}$ are equivalent norms. □

In the following definition, we introduce a particular form of the function p that will be of use in our investigation.

Definition 3.2 *Let $k, r : [a,\infty)_{\mathbb{T}} \to (0,\infty)$ be rd–continuous. For $t \in [a,\infty)_{\mathbb{T}}$ we define*

$$P(t,s) := \int_s^{\sigma(t)} \frac{1}{r(u)} \Delta u, \quad a \leq s \leq \sigma(t); \tag{3.7}$$

and for any constant $K > 1$, define

$$p(t) := K[P(t,a)+1]k(t), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.8}$$

As a special case, consider $r \equiv 1$ in (3.7). Then (3.7) becomes

$$P(t,s) = \sigma(t) - s, \quad a \leq s \leq \sigma(t);$$

and (3.8) becomes

$$p(t) = K[\sigma(t) - s + 1]k(t), \quad t \in [a,\infty)_{\mathbb{T}}.$$

For our investigation of (1.1) and (1.2) the function r in the preceding context, of course, is determined from (1.1) or (1.2), while k will appear from the following generalised Lipschitz condition on F

$$|F(t,u,z) - F(t,v,w)| \leq k(t)[|u-v| + |z-w|], \quad t \in [a,\infty)_{\mathbb{T}},\ (u,v,w,z) \in \mathbb{R}^4;$$

or the special case

$$|F(t,u) - F(t,v)| \leq k(t)|u-v|, \quad t \in [a,\infty)_{\mathbb{T}},\ (u,v) \in \mathbb{R}^2.$$

Our analysis of (1.1) (and (1.2)) will involve the existence and uniqueness of fixed–points to certain delta integral operators. We now state an equivalence result between solutions to the dynamic equation (1.1) and a particular delta integral equation.

Lemma 3.3 *Consider* (1.1). *Let* $r : [a,\infty)_{\mathbb{T}} \to (0,\infty)$ *be rd–continuous and let* $F : [a,\infty)_{\mathbb{T}} \times \mathbb{R}^2 \to \mathbb{R}$ *be rd–continuous. Let* P *be defined in* (3.7). *If there is a constant* $M > 0$ *such that for all* $x \in \mathcal{X}_M$

$$\int_a^\infty P(t,a)F(t,x^\sigma(t),x^{\Delta\sigma}(s))\ \Delta t \leq M, \quad t \in [a,\infty)_{\mathbb{T}}; \tag{3.9}$$

then the dynamic equation (1.1) *has a solution with* $\lim_{t\to\infty} x(t) = M$ *and* $\lim_{t\to\infty} r(t)x^\Delta(t) = 0$ *if and only if the delta integral equation*

$$x(t) = M - \int_t^\infty P(s,t)F(s,x^\sigma(s),x^{\Delta\sigma}(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}; \tag{3.10}$$

has a solution on $[a,\infty)_{\mathbb{T}}$.

Proof Assume the dynamic equation (1.1) has a solution x with $\lim_{t\to\infty} x(t) = M$ and $\lim_{t\to\infty} r(t)x^\Delta(t) = 0$. Then

$$[r(t)x^\Delta(t)]^\Delta = -F(t,x^\sigma(t),x^{\Delta\sigma}(t)), \quad t \in [a,\infty)_{\mathbb{T}}.$$

Taking the delta integral on both sides from t to ∞ we obtain

$$r(t)x^\Delta(t) = \int_t^\infty F(s,x^\sigma(s),x^{\Delta\sigma}(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.$$

Dividing by $r(t)$ and integrating again we have

$$M - x(t) = \int_t^\infty \frac{1}{r(s)} \int_s^\infty F(u,x^\sigma(u),x^{\Delta\sigma}(u))\ \Delta u\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.$$

Integrating by parts and using the definition of $P(t,s)$ in (3.7) we obtain

$$x(t) = M - \int_t^\infty P(s,t)F(s, x^\sigma(s), x^{\Delta\sigma}(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.$$

Conversely, assume x is a solution of (3.10) on $[a,\infty)_{\mathbb{T}}$, that is

$$x(t) = M - \int_t^\infty P(s,t)F(s, x^\sigma(s), x^{\Delta\sigma}(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.$$

Note that $\lim_{t\to\infty} x(t) = M$. Taking the delta derivative we obtain

$$\begin{aligned} x^\Delta(t) &= P(t,\sigma(t))F(t, x^\sigma(t), x^{\Delta\sigma}(t)) + \frac{1}{r(t)}\int_t^\infty F(s, x^\sigma(s), x^{\Delta\sigma}(s))\ \Delta s, \qquad t \in [a,\infty)_{\mathbb{T}}; \\ &= \frac{1}{r(t)}\int_t^\infty F(s, x^\sigma(s), x^{\ \Delta\sigma}(s))\ \Delta s. \end{aligned}$$

Therefore

$$r(t)x^\Delta(t) = \int_t^\infty F(s, x^\sigma(s), x^{\Delta\sigma}(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}}; \tag{3.11}$$

which implies that

$$\lim_{t\to\infty} r(t)x^\Delta(t) = 0.$$

Differentiating both sides of (3.11) we obtain the desired result

$$[r(t)x^\Delta(t)]^\Delta = -F(t, x^\sigma(t), x^{\Delta\sigma}(t)), \quad t \in [a,\infty)_{\mathbb{T}}.$$

□

4 Main Results

In this section we sate and prove our main results which will be obtained by use of the Banach fixed–point theorem. Banach's fixed–point theorem is one of the most basic, yet most compelling, ideas from fixed–point theory - mainly because the theorem produces a wide range of qualitative *and* quantitative information about solutions.

Let $(Y, \|\cdot\|_Y)$ be a Banach space and $T : Y \to Y$. The map T is said to be contractive if there exists a positive constant $\alpha < 1$ such that

$$\|T(x) - T(y)\|_Y \le \alpha\|x - y\|_Y, \quad \forall x, y \in Y.$$

The constant α is called the contraction constant of T.

For any given $y \in Y$ we define the sequence $\{T^i(y)\}$ recursively by: $T^0(y) := y$; and $T^{i+1}(y) := T(T^i(y))$.

Theorem 4.1 (Banach) *Let $(Y, \|\cdot\|_Y)$ be a Banach space and let $T : Y \to Y$ be contractive. Then T has a unique fixed–point u and $T^i(y) \to u$ for each $y \in Y$. If we start at an arbitrary $y \in Y$ then the following estimate on the "error" between the ith iteration T^iy and the fixed point u, holds*

$$\|F^iy - u\|_Y \le \frac{\alpha^i}{1-\alpha}\|y - Fy\|_Y. \tag{4.1}$$

Proof See [8, p.10] or [16, Theorem 7.5]. □

The following two results furnish the existence of a unique, positive solution to (1.2) and (1.1) for solutions that possess a horizontal asymptote.

Theorem 4.2 *Consider* (1.2). *Let* $r, k : [a,\infty)_{\mathbb{T}} \to (0,\infty)$ *be rd–continuous and let* $F : [a,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *be rd–continuous. Let* P *and* p *be defined as in Definition 3.2 with* k *and* F *satisfying*

$$F(t,u) \geq 0, \qquad \forall (t,u) \in [a,\infty)_{\mathbb{T}} \times [0,\infty); \tag{4.2}$$

$$|F(t,u) - F(t,v)| \leq k(t)|u-v|, \quad t \in [a,\infty)_{\mathbb{T}},\ (u,v) \in \mathbb{R}^2; \tag{4.3}$$

$$\int_a^\infty p(t)\ \Delta t < \infty. \tag{4.4}$$

Let $M > 0$ *be a constant such that for all* $x \in \mathcal{X}_{\mathcal{M}}$

$$\int_a^\infty P(t,a)F(t,x^\sigma(t))\ \Delta t \leq M, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.5}$$

Then there is a unique positive solution $x \in \mathcal{X}_M$ *of* (1.2) *satisfying* $\lim_{t\to\infty} x(t) = M$. *Furthermore*

$$x(t) \geq r(t)x^\Delta(t) \int_a^t \frac{1}{r(u)}\ \Delta u, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.6}$$

In addition, if $x_0 \in \mathcal{X}_M$ *and we define the "Picard iterates"* x_i *recursively by*

$$x_{i+1}(t) := M - \int_t^\infty P(s,t)F(s,x_i^\sigma(s))\ \Delta s, \quad t \in [a,\infty)_{\mathbb{T}};$$

then $\lim_{i\to\infty} x_i(t) = x(t)$ *uniformly on* $[a,\infty)_{\mathbb{T}}$.

Proof Consider $\mathcal{X}_M$ defined in (3.2). Let $w(t) := e_{\ominus p}(t,a)$ for all $t \in [a,\infty)_{\mathbb{T}}$ where $p(t) := K[P(t,a)+1]k(t)$ and $K > 1$ is an arbitrary constant. Consider the pair $(\mathcal{X}_M, \|\cdot\|_w)$, which forms a Banach space by Lemma 3.1. Define an operator T on $\mathcal{X}_M$ by

$$[Tx](t) := M - \int_t^\infty P(s,t)F(s,x^\sigma(s))\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{4.7}$$

We will now use Banach's fixed–point theorem to show that T has a unique fixed point in $\mathcal{X}_M$. If $x \in \mathcal{X}_M$, then by (4.2), $F(t,x^\sigma(t)) \geq 0$ and hence

$$0 \leq [Tx](t) \leq M, \quad t \in [a,\infty)_{\mathbb{T}}.$$

Also, for all $t \in [a,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
[Tx]^\Delta(t) &= P(t,\sigma(t))F(t,x^\sigma(t)) + \frac{1}{r(t)}\int_t^\infty F(s,x^\sigma(s))\ \Delta s \\
&= \frac{1}{r(t)}\int_t^\infty F(s,x^\sigma(s))\ \Delta s \\
&\geq 0.
\end{aligned}$$

Since $Tx : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ and $(Tx)^\Delta : [a,\infty)_{\mathbb{T}} \to \mathbb{R}$ are continuous we have that $T : \mathcal{X}_M \to \mathcal{X}_M$.

We now show that T is a contraction mapping on $\mathcal{X}_M$. For $x, y \in \mathcal{X}_M$, consider

$$\begin{aligned}
\frac{|[Tx](t) - [Ty](t)|}{w(t)} &\leq \frac{1}{w(t)} \int_t^\infty P(s,t)|F(s,x^\sigma(s)) - F(s,y^\sigma(s))|\ \Delta s \\
&\leq \frac{1}{w(t)} \int_t^\infty P(s,t)k(s)|x^\sigma(s) - y^\sigma(s)|\ \Delta s \\
&= \frac{1}{w(t)} \int_t^\infty P(s,t)k(s)w^\sigma(s)\frac{|x^\sigma(s) - y^\sigma(s)|}{w^\sigma(s)}\ \Delta s \\
&\leq \|x-y\|_w \frac{1}{w(t)} \int_t^\infty P(s,t)k(s)w^\sigma(s)\ \Delta s.
\end{aligned}$$

Using (3.5) we obtain

$$k(t)w^\sigma(t) = -\frac{w^\Delta(t)}{K[P(t,a)+1]}, \quad t \in [a,\infty)_{\mathbb{T}}.$$

Therefore, from (4.8), we obtain, for $t \in [a,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
\frac{|[Tx](t) - [Ty](t)|}{w(t)} &\leq -\frac{\|x-y\|_w}{Kw(t)} \int_t^\infty \frac{P(s,t)w^\Delta(s)}{P(s,a)+1}\ \Delta s \\
&\leq -\frac{\|x-y\|_w}{Kw(t)} \int_t^\infty w^\Delta(s)\Delta s \|x-y\|_w \\
&= \frac{w(t) - L}{Kw(t)} \|x-y\|_w \\
&\leq \frac{1}{K}\|x-y\|_w.
\end{aligned}$$

Taking the sup over t we obtain

$$\|Tx - Ty\|_w \leq \frac{1}{K}\|x-y\|_w = \alpha\|x-y\|_w;$$

where $\alpha := 1/K < 1$, and hence T is a contraction mapping on $\mathcal{X}_M$. Therefore, from Banach's fixed–point theorem, T has a unique fixed point $x \in \mathcal{X}_M$ and so by Lemma 3.3 there is a unique positive solution of (1.2) in $\mathcal{X}_M$ satisfying $\lim_{t\to\infty} x(t) = M$.

To see that (4.6) holds note that the solution x satisfies, for each $t \in [a,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
x(t) &= M - \int_t^\infty P(s,t)F(s,x^\sigma(s))\ \Delta s \\
&= M - \int_t^\infty \left[P(s,a) - \int_a^t \frac{1}{r(u)}\Delta u\right] F(s,x^\sigma(s))\ \Delta s \\
&= M - \int_t^\infty P(s,a)F(s,x^\sigma(s))\ \Delta s + \int_a^t \frac{1}{r(u)}\Delta u \int_t^\infty F(s,x^\sigma(s))\ \Delta s \\
&\geq M - \int_a^\infty P(s,a)F(s,x^\sigma(s))\ \Delta s + \int_a^t \frac{1}{r(u)}\Delta u \int_t^\infty F(s,x^\sigma(s))\ \Delta s \\
&\geq \int_a^t \frac{1}{r(u)}\Delta u \int_t^\infty F(s,x^\sigma(s))\ \Delta s.
\end{aligned}$$

Since

$$x^{\Delta}(t) = \frac{1}{r(t)} \int_t^{\infty} F(s, x^{\sigma}(s))\Delta s, \quad t \in [a, \infty)_{\mathbb{T}};$$

we see that (4.6) holds. □

Theorem 4.3 *Consider* (1.1). *Let* $r, k : [a, \infty)_{\mathbb{T}} \to (0, \infty)$ *be rd–continuous and let* $F : [a, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *be rd–continuous. Let* $K > 1$ *and* R *be positive constants with:* $r > R$ *on* $[a, \infty)_{\mathbb{T}}$*;* $KR > 1$*. For* P *and* p *defined as in Definition 3.2, let* k *and* F *satisfy*

$$F(t, u, w) \geq 0, \qquad \forall t \in [a, \infty)_{\mathbb{T}}, \ (u, w) \in \mathbb{R}^2; \tag{4.8}$$

$$|F(t, u, w) - F(t, v, z)| \leq k(t)\left(|u - v| + |w - z|\right), \quad t \in [a, \infty)_{\mathbb{T}}; \ (u, v) \in \mathbb{R}^2; \tag{4.9}$$

$$\int_a^{\infty} p(t) \ \Delta t < \infty. \tag{4.10}$$

Let $M > 0$ *be a constant such that for all* $x \in \mathcal{X}_{\mathcal{M}}$

$$\int_a^{\infty} P(t, a) F(t, x^{\sigma}(t), x^{\Delta\sigma}(t)) \ \Delta t \leq M, \quad t \in [a, \infty)_{\mathbb{T}}. \tag{4.11}$$

Then there is a unique positive solution $x \in \mathcal{X}_M$ *of* (1.1) *satisfying* $\lim_{t \to \infty} x(t) = M$*. Furthermore* (4.6) *holds. In addition, if* $x_0 \in \mathcal{X}_M$ *and we define the "Picard iterates"* x_i *recursively by*

$$x_{i+1}(t) := M - \int_t^{\infty} P(s, t) F(s, x_i^{\sigma}(s), x_i^{\Delta\sigma}(s))\Delta s, \quad t \in [a, \infty)_{\mathbb{T}};$$

then $\lim_{i \to \infty} x_i(t) = x(t)$ *uniformly on* $[a, \infty)_{\mathbb{T}}$.

Proof Much of this proof is similar to the proof of Theorem 4.2 and so we only sketch the proof. Define T on $\mathcal{X}_M$ by

$$[Tx](t) = M - \int_t^{\infty} P(s, t) F(s, x^{\sigma}(s), x^{\Delta\sigma}(s))\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}.$$

It follows that $T : \mathcal{X}_M \to \mathcal{X}_M$. We want to show T is a contraction mapping on $\mathcal{X}_M$. As in the proof of Theorem 4.2 we get for $x, y \in \mathcal{X}_M$

$$\frac{|[Tx](t) - [Ty](t)|}{w(t)} \leq \frac{1}{K} \|x - y\|_w.$$

Now for $x, y \in \mathcal{X}_M$, consider, for $t \in [a,\infty)_{\mathbb{T}}$,

$$
\begin{aligned}
& \frac{|(Tx)^{\Delta}(t) - (Ty)^{\Delta}(t)|}{w(t)} \\
\leq\ & \frac{1}{r(t)w(t)} \int_t^{\infty} |F(s, x^{\sigma}(s), x^{\Delta\sigma}(s)) - F(s, y^{\sigma}(s), y^{\Delta\sigma}(s))|\ \Delta s \\
\leq\ & \frac{1}{r(t)w(t)} \int_t^{\infty} k(s) \left[|x^{\sigma}(s) - y^{\sigma}(s)| + |x^{\Delta\sigma}(s)) - y^{\Delta\sigma}(s))|\right]\ \Delta s \\
\leq\ & \frac{1}{r(t)w(t)} \int_t^{\infty} k(s)w^{\sigma}(s) \left[\frac{|x^{\sigma}(s) - y^{\sigma}(s)|}{w^{\sigma}(s)} + \frac{|x^{\Delta\sigma}(s)) - y^{\Delta\sigma}(s))}{w^{\sigma}(s)}|\right]\ \Delta s \\
\leq\ & \|x-y\|_w \frac{1}{r(t)w(t)} \int_t^{\infty} k(s)w^{\sigma}(s)\ \Delta s \\
=\ & \|x-y\|_w \frac{1}{r(t)w(t)} \int_t^{\infty} \frac{-w^{\Delta}(s)}{K[P(s,a)+1]}\ \Delta s \\
\leq\ & -\frac{\|x-y\|_w}{Kr(t)w(t)} \int_t^{\infty} w^{\Delta}(s)\ \Delta s \\
=\ & \frac{1}{KR} \frac{w(t)-L}{w(t)} \|x-y\|_w \\
\leq\ & \frac{1}{KR} \|x-y\|_w.
\end{aligned}
$$

It follows that

$$\|Tx - Ty\|_w \leq \alpha \|x-y\|_w;$$

where

$$\alpha := \max\left\{\frac{1}{K}, \frac{1}{KR}\right\} < 1.$$

□

The following result bounds the delta derivative of solutions furnished by the previous two theorems.

Corollary 4.4 *In Theorems 4.2 and 4.3 if we replace* $\mathcal{X}_M$ *by*

$$\mathcal{Y}_M := \{x \in \mathcal{X}_M : x^{\Delta}(t) \leq M,\ \ t \in [a,\infty)_{\mathbb{T}}\}$$

and assume that the inequalities

$$
\begin{aligned}
\frac{1}{r(t)} \int_a^{\infty} F(t, x^{\sigma}(t), x^{\Delta\,\sigma}(t))\ \Delta t \leq M, \quad t \in [a,\infty)_{\mathbb{T}}; \\
\frac{1}{r(t)} \int_a^{\infty} F(t, x^{\sigma}(t), x^{\Delta\,\sigma}(t))\ \Delta t \leq M, \quad t \in [a,\infty)_{\mathbb{T}};
\end{aligned}
$$

hold for all $x \in \mathcal{Y}_M$*, then* (1.1) *(resp.* (1.2)*) has a unique positive solution in* $\mathcal{Y}_M$*. In particular,* $0 \leq x^{\Delta} \leq M$ *on* $[a,\infty)_{\mathbb{T}}$.

In the following result, we obtain the existence and uniqueness of "eventually positive" solutions.

Theorem 4.5 *Consider* (1.1). *Let* $r, k : [a,\infty)_{\mathbb{T}} \to (0,\infty)$ *be rd–continuous and let* $F : [a,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *be rd–continuous. Let* P *be defined as in Definition 3.2 with* $\int_a^\infty P(s,a)k(s)\Delta s < \infty$. *Assume* F *satisfies* (4.9) *with*

$$\frac{1}{r(t)} \int_t^\infty k(s)\ \Delta s < 1, \quad \textit{for sufficiently large } t.$$

Let the constant $M > 0$ *be given such that there is a* $v \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ *so that the two integrals*

$$\int_a^\infty P(s,a)|F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s, \quad \int_a^\infty |F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s;$$

are finite. Then there is a unique solution $x \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ *such that for some* $t_0 \in [a,\infty)_{\mathbb{T}}$, $x > 0$ *on* $[a,\infty)_{\mathbb{T}} \cap [t_0,\infty)$ *and* $\lim_{t\to\infty} x(t) = M$. *Furthermore,* (4.6) *holds on* $[a,\infty)_{\mathbb{T}} \cap [t_0,\infty)$.

Proof Let $M > 0$ be given and define T_M on $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ by

$$[T_M x](t) = M - \int_t^\infty P(s,t)F(s, x^\sigma(s), x^{\Delta\sigma}(s))\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.$$

To see that T_M is well defined, let $x \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ and consider

$$\begin{aligned}
0 &\le \int_a^\infty P(s,t)|F(s, x^\sigma(s), x^{\Delta\sigma}(s))|\ \Delta s \\
&\le \int_a^\infty P(s,t)|F(s, x^\sigma(s), x^{\Delta\sigma}(s)) - F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s \\
&+ \int_a^\infty P(s,t)|F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s \\
&\le \|x - v\|_1^{\mathbb{T}} \int_a^\infty P(s,t)k(s)\ \Delta s + \int_a^\infty P(s,t)|F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s \\
&\le \|x - v\|_1^{\mathbb{T}} \int_a^\infty P(s,a)k(s)\ \Delta s + \int_a^\infty P(s,a)|F(s, x^\sigma(s), x^{\Delta\sigma}(s))|\ \Delta s \\
&< \infty.
\end{aligned}$$

Therefore,

$$\int_a^\infty P(s,t)|F(s, x^\sigma(s), x^{\Delta\sigma}(s))|\ \Delta s < \infty, \quad \text{for all } x \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}),$$

and T_M is well defined on $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$. Clearly, $T_M(x)$ is continuous on $[a,\infty)_{\mathbb{T}}$. Also for $x \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$,

$$\begin{aligned}
&\int_a^\infty |F(s, x^\sigma(s), x^{\Delta\sigma}(s))|\ \Delta s \\
&\le \int_a^\infty |F(s, x^\sigma(s), x^{\Delta\sigma}(s)) - F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s + \int_a^\infty |F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s \\
&\le \|x - v\|_1 \int_a^\infty k(s)\ \Delta s + \int_a^\infty |F(s, v^\sigma(s), v^{\Delta\sigma}(s))|\ \Delta s \\
&< \infty.
\end{aligned}$$

Hence

$$[T_M x]^\Delta(t) = \frac{1}{r(t)} \int_a^\infty F(s, x^\sigma(s), x^{\Delta\sigma}(s))\ \Delta s;$$

exists and is continuous on $[a,\infty)_{\mathbb{T}}$. Thus $T_M : C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}}) \to C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$.

Next we show T_M is a contraction mapping on a subset of $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$, where our norm $\|\cdot\|_1$ is given by (3.1). To this end let $x, y \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ and consider

$$\begin{aligned} |[T_M x](t) - [T_M y](t)| &\leq \int_t^\infty P(s,t)|F(s, x^\sigma(s), x^{\Delta\sigma}(s)) - F(s, y^\sigma(s), y^{\Delta\sigma}(s))|\ \Delta s \\ &\leq \int_t^\infty P(s,t)k(s)[|x^\sigma(s) - y^\sigma(s)| + |x^{\Delta\sigma}(s) - y^{\Delta\sigma}(s)|]\ \Delta s \\ &\leq \|x - y\|_1 \int_t^\infty P(s,t)k(s)\ \Delta s. \end{aligned} \tag{4.12}$$

Also, for $x, y \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$, consider

$$\begin{aligned} |[T_M x]^\Delta(t) - [T_M y]^\Delta(t)| &\leq \frac{1}{r(t)} \int_t^\infty |F(s, x^\sigma(s), x^{\Delta\sigma}(s)) - F(s, y^\sigma(s), y^{\Delta\sigma}(s))|\ \Delta s \\ &\leq \frac{1}{r(t)} \int_t^\infty k(s)[|x^\sigma(s) - y^\sigma(s)| + |x^{\Delta\sigma} - y^{\Delta\sigma}|\ \Delta s \\ &\leq \|x - y\|_1 \frac{1}{r(t)} \int_t^\infty k(s)\ \Delta s. \end{aligned} \tag{4.13}$$

Now we choose $t_1 \in [a,\infty)_{\mathbb{T}}$, sufficiently large so that:

$$\int_t^\infty P(s,t)\Delta s \leq \alpha < 1; \quad \frac{1}{r(t)} \int_t^\infty k(s)\Delta s \leq \alpha < 1, \quad \text{for} \quad t \in [a,\infty)_{\mathbb{T}} \cap [t_1,\infty).$$

Then using (4.12) and (4.13) we see that T_M is a contraction mapping on $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}} \cap [t_1,\infty))$. By Banach's fixed–point theorem we conclude that T_M has a unique fixed point in $C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}} \cap [t_1,\infty))$ and this leads to a solution on all of $[a,\infty)_{\mathbb{T}}$ by the assumption that initial value problems for (1.2) have unique solutions. □

We present our final result, which concerns non–negative solutions.

Theorem 4.6 *Consider* (1.2). *Let* $r, k : [a,\infty)_{\mathbb{T}} \to (0,\infty)$ *be rd–continuous and let* $F : [a,\infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ *be rd–continuous. Let* P *be defined as in Definition 3.2. Assume* F *satisfies* (4.2) *and the Lipschitz condition* (4.3) *with*

$$\int_a^\infty P(s,a)k(s)\ \Delta s < 1. \tag{4.14}$$

If there exists a $v \in C^1_{\mathbb{T}}([a,\infty)_{\mathbb{T}})$ *such that*

$$\int_a^\infty P(s,t)F(s, v^\sigma(s))\ \Delta s < \infty; \tag{4.15}$$

then there is a unique solution x *of* (1.2) *satisfying*

$$x(a) = 0; \quad x(t) \geq 0, \quad \textit{for} \quad t \in [a,\infty)_{\mathbb{T}};$$

with

$$\lim_{t\to\infty} x(t) = \int_a^\infty P(s,a)F(s,x^\sigma(s))\ \Delta s.$$

Furthermore, x satisfies (4.6)*.*

Proof Consider the Banach space $(\mathcal{H}, \|\cdot\|_0)$ defined in Section 3. We then define $\hat{T}$ on $\mathcal{H}$ by

$$[\hat{T}x](t) := \int_a^\infty P(s,a)F(s,x^\sigma(s))\Delta s - \int_t^\infty P(s,t)F(s,x^\sigma(s))\Delta s, \quad t \in [a,\infty)_{\mathbb{T}}.$$

It follows similar to the proof of Theorem 4.5 that the assumption (4.15) ensures $\hat{T}$ is well defined on $\mathcal{H}$ and $\hat{T}x$ is continuous on $[a,\infty)_{\mathbb{T}}$. Furthermore, $[\hat{T}x](a) = 0$.

Next, note that

$$[\hat{T}x]^\Delta(t) = \frac{1}{r(t)}\int_t^\infty F(s,x^\sigma(s))\ \Delta s > 0, \quad t \in [a,\infty)_{\mathbb{T}};$$

so $\hat{T}x$ is increasing with

$$\lim_{t\to\infty}[\hat{T}x](t) = \int_a^\infty P(s,a)F(s,x^\sigma(s))\Delta s < \infty$$

and hence $[\hat{T}x](t) \geq 0$ and is bounded on $[a,\infty)_{\mathbb{T}}$. Therefore $\hat{T} : \mathcal{H} \to \mathcal{H}$.

We now find a different expression for $[\hat{T}x](t)$, namely

$$\begin{aligned}
[\hat{T}x](t) &= \int_a^\infty P(s,a)F(s,x^\sigma(s))\ \Delta s - \int_t^\infty P(s,t)F(s,x^\sigma(s))\ \Delta s \\
&= \int_a^t P(s,a)F(s,x^\sigma(s))\ \Delta s + \int_t^\infty [P(s,a) - P(s,t)]F(s,x^\sigma(s))\ \Delta s \\
&= \int_a^t P(s,a)F(s,x^\sigma(s))\ \Delta s + \int_a^t \frac{1}{r(u)}\ \Delta u \int_t^\infty F(s,x^\sigma(s))\ \Delta s.
\end{aligned}$$

To show that $\hat{T}$ is a contraction mapping on $\mathcal{H}$, let $x, y \in \mathcal{H}$ and consider

$$\begin{aligned}
&|[\hat{T}x](t) - [\hat{T}y](t)| \\
\leq\ & \int_a^t P(s,a)|F(s,x^\sigma(s)) - F(s,y^\sigma(s))|\ \Delta s + \int_a^t \frac{1}{r(u)}\ \Delta u \int_t^\infty |F(s,x^\sigma(s)) - F(s,y^\sigma(s))|\ \Delta s \\
\leq\ & \int_a^t P(s,a)k(s)|x^\sigma(s)) - y^\sigma(s)|\ \Delta s + \int_a^t \frac{1}{r(u)}\Delta u \int_t^\infty k(s)|x^\sigma(s) - y^\sigma(s)|\ \Delta s \\
\leq\ & \|x-y\|_0 \int_a^t P(s,a)k(s)\ \Delta s + \|x-y\|_0 \int_t^\infty \left(\int_a^t \frac{1}{r(u)}\Delta\right) uk(s)\ \Delta s \\
\leq\ & \|x-y\|_0 \int_a^\infty P(s,a)k(s)\ \Delta s = \alpha\|x-y\|_0,
\end{aligned}$$

where

$$\alpha := \int_a^\infty P(s,a)k(s)\ \Delta s < 1$$

by (4.14). Hence $\hat{T}$ is a contraction mapping on $\mathcal{H}$.

□

5 Examples

We now present some simple examples to illustrate the application of our new theorems and how they advance existing knowledge.

Example 5.1 *Consider the time scale* $\mathbb{T} = \mathbb{R}$ *and the ordinary differential equation*

$$\left(\frac{1}{t^\delta}x'\right)' + \frac{1}{t^\gamma}\frac{x}{1+x^2} = 0, \quad t \in [1, \infty);$$

where $\gamma > \delta + 2 > 1$ *are constants.*

For this problem we can take: $k(t) = \frac{1}{t^\gamma}$*;* $P(t,1) = \left(t^{\delta+1} - 1\right)/(\delta+1)$*, and* $p(t) = 3[P(t,1)+1]k(t)$ *and show that the hypotheses of Theorem 4.2 hold.*

We note that the theorems in [12, 21] do not directly apply to the previous example.

Example 5.2 *For an arbitrary, unbounded time scale interval of the form* $[a, \infty)_{\mathbb{T}}$*,* $a > 0$*, we get a result for dynamic equations of the form*

$$\left((\sqrt{t} + \sqrt{\sigma(t)})x^\Delta\right)^\Delta + F(t, x^\sigma) = 0, \tag{5.1}$$

where

$$\left|\frac{\partial F}{\partial u}(t,u)\right| \le \frac{M}{t^\beta}, \quad t \in [a,\infty)_{\mathbb{T}}, \ \ u \in \mathbb{R},$$

where $\beta > \frac{3}{2}$ *and* F *satisfies the hypotheses of Theorem 4.2.*

No existing results apply to the previous example.

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