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DISCRETE LINEAR HAMILTONIAN SYSTEMS: A SURVEY

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ABSTRACT. We present a survey on recent results connected to linear Hamiltonian difference systems. In order to obtain unified results on continuous and discrete Hamiltonian systems we also consider a approach via so-called time scales.

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1. INTRODUCTION

In this paper we report an up-to-date account of the theory of symplectic difference systems. Such systems include as a special case linear Hamiltonian difference systems, which in turn include as special cases Sturm–Liouville difference equations of order $2n$, $n \in \mathbb{N}$, and self–adjoint vector difference equations, which include as a special case Sturm–Liouville difference equations of second order. The concept of generalized zeros of solutions of symplectic difference systems, and hence disconjugacy of these systems, is introduced. Oscillatory behavior of symplectic difference systems is addressed, and in this context so–called trigonometric systems, together with transformations which transform any symplectic system into a trigonometric system without changing the oscillatory behavior of the original system, are considered. As in the classical "continuous" case, the concept of disconjugacy is very useful in establishing the discrete calculus of variations. For this theory, we shall present discrete versions of the Euler–Lagrange condition, the Legendre condition, the strengthened Legendre condition, and the strengthened Jacobi condition.

The paper is organized as follows. Section 2 introduces linear Hamiltonian difference systems and explains how they are special cases of symplectic difference systems

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and in what way Sturm–Liouville difference equations of higher order are special cases of them. Section 3 gives an introduction to symplectic systems and introduces important notions such as conjoined bases, generalized zeros, focal points, and disconjugacy. It also features a so–called Reid roundabout theorem, which gives conditions that are equivalent to disconjugacy, among them a discrete version of the strengthened Jacobi condition and solvability of a Riccati equation subject to certain conditions. Next, in Section 4, we introduce the concept and provide the motivation of trigonometric systems. Matrix solutions of such trigonometric systems satisfy several identities, which in the scalar case reduce to well-known equalities. We also offer an explicit transformation which transforms any symplectic system into a trigonometric system and preserves oscillatory behavior. This theorem, together with oscillatory behavior of trigonometric systems, can be used to study the oscillatory behavior of arbitrary symplectic systems. Section 5 gives the so–called reduction of order theorem which allows representation of every solution in terms of a certain single conjoined solution of the symplectic system. This leads to a discussion of recessive and dominant solutions, which are introduced and studied here. Next, Section 6 presents an overview of the discrete calculus of variations. It introduces the Euler–Lagrange equations and gives necessary conditions for a local minimum of discrete variational problems in terms of the first and second variations. We emphasize that the discrete Legendre condition appears to be very different from its "continuous" counterpart and suggest the following new (previously unpublished; see Remark 6.5 below) approach: We consider a certain discrete variational problem with step size h instead of 1 and derive a Legendre condition for such a problem. This condition reduces to the Legendre condition for the discrete case if $h = 1$, and as $h \to 0$, it becomes the Legendre condititon for the continuous case. This technique is strongly connected to the theory that we present in our final Section 7: The so–called theory of measure chains (or time scales) has been created by Stefan Hilger in order to unify discrete and continuous analysis. We give an introduction to this theory and present some results connected to Sturm–Liouville equations and linear Hamiltonian systems. Work in this area is still in progress.

Most of the results presented in this paper are modeled from the recent book by C. Ahlbrandt and A. Peterson on "Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations" [8]. Further references that especially address the examples discussed in Section 2 include [7, 9, 14, 22, 23, 27, 28, 29, 30, 31, 32, 37, 38, 40, 43, 44, 45, 46, 47, 48]. Articles [11, 13, 15, 16, 17, 19, 20] supplement the material of Section 3, while [10, 21] are additional references for Section 4. For Section 5, the reader may also see [4, 6], whereas [5, 41] supplement the material presented in Section 6. Finally, introduction to the theory of time scales (which is considered in Section 7) is given in [12, 33, 39]. Further results in this rather new and fast developing area of research include [1, 2, 3, 24, 25, 26, 34, 35, 36, 42].

2. EXAMPLES LEADING TO HAMILTONIAN SYSTEMS

In this paper we are mainly concerned with the discrete linear vector Hamiltonian system

$$\Delta x(t) = A(t)x(t+1) + B(t)u(t), \quad \Delta u(t) = C(t)x(t+1) - A^*(t)u(t). \tag{2.1}$$

Here we shall assume that $B(t)$ and $C(t)$ are $n \times n$ Hermitian matrix functions on the discrete interval $[a, b] := \{a, a+1, ..., b\}$, where $a \leq b$ are integers, and we assume that $I - A(t)$ is nonsingular on $[a, b]$. With these assumptions, solutions x, u of initial value problems for (2.1) exist on the discrete interval $[a, b+1]$. In this section we will discuss several important examples which lead to Hamiltonian systems. Solving (2.1) for $x(t+1)$ and $u(t+1)$ we get the symplectic system

$$x(t+1) = E(t)x(t) + F(t)u(t), \quad u(t+1) = G(t)x(t) + H(t)u(t) \tag{2.2}$$

where

$$E(t) = [I - A(t)]^{-1}, \;\; F(t) = E(t)B(t),$$
$$G(t) = C(t)E(t), \;\; H(t) = I - A^*(t) + C(t)E(t)B(t).$$

The system (2.2) is called symplectic because if

$$\mathcal{S}(t) := \begin{bmatrix} E(t) & F(t) \\ G(t) & H(t) \end{bmatrix},$$

then $\mathcal{S}(t)$ is a symplectic matrix for $t \in [a, b]$. Recall that a $2n \times 2n$ constant matrix M is said to be *symplectic* provided

$$M^* \mathcal{J} M = \mathcal{J},$$

where the $2n \times 2n$ constant matrix $\mathcal{J}$ is defined by

$$\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We start with the simplest example that leads to a Hamiltonian system.

Example 2.1. Consider the second order linear self–adjoint difference equation

$$\Delta[p(t)\Delta y(t-1)] + q(t)y(t) = 0, \tag{2.3}$$

where t is in the discrete interval $[a+1, b+1]$. We assume $p(t) \neq 0$ is real valued on the discrete interval $[a+1, b+2]$, and $q(t)$ is a real valued function on $[a+1, b+1]$. If we set

$$x(t) = y(t-1), \;\; u(t) = p(t)\Delta y(t-1),$$

then we get the symplectic system

$$x(t+1) = x(t) + \frac{1}{p(t)}u(t), \quad u(t+1) = -q(t)x(t) + \left[1 - \frac{q(t)}{p(t)}\right]u(t)$$

and the equivalent Hamiltonian system

$$\Delta x(t) = 0 \cdot x(t+1) + \frac{1}{p(t)}u(t), \quad \Delta u(t) = -q(t)x(t+1) - 0 \cdot u(t).$$

Later we will see that symplectic systems and Hamiltonian systems are equivalent under certain assumptions.

Equation (2.3) has been studied in a large number of papers. For a good introduction to the study of equation (2.3) see Chapter 6 in [40] or Chapter 1 in [8]. The self–adjoint equation (2.3) also arises in the discrete calculus of variations. An important concept in the study of the second order self–adjoint difference equation (2.3) is the notion of a generalized zero which was first introduced for difference equations by Hartman [32]. The definition given here is a slight generalization as we are not assuming $p(t) > 0$ on $[a+1, b+2]$.

Definition 2.2. We say that a nontrivial solution y of (2.3) has a *generalized zero* at a if and only if $y(a) = 0$ and a generalized zero at $t_0 > a$ provided $y(t_0) = 0$ or if $p(t_0)y(t_0 - 1)y(t_0) < 0$.

Note that with this definition every solution of the Fibonacci difference equation, expressed in self–adjoint form,

$$\Delta[(-1)^t \Delta y(t-1)] + (-1)^t y(t) = 0$$

has infinitely many generalized zeros on the discrete interval $[0, \infty)$.

Definition 2.3. We say that the self–adjoint difference equation (2.3) is *disconjugate* on $[a, b+2]$ provided no nontrivial solution has two generalized zeros in $[a, b+2]$.

In the study of (2.3), disconjugacy of (2.3) plays a very important role.

We now give a relationship between symplectic systems and Hamiltonian systems (see [8, pp. 82–85]).

Theorem 2.4. *Every Hamiltonian system* (2.1) (*with* $I - A(t)$ *nonsingular*) *is equivalent to a sympletic system* (2.2) *with* $E(t)$ *nonsingular. We pointed out earlier that every Hamiltonian system can be written as a symplectic system with* $E(t) = [I - A(t)]^{-1}$, *which is nonsingular. Conversely, if* $E(t)$ *is assumed to be nonsingular it is proved in* [8, p. 83] *that any solution of the symplectic system* (2.2) *is a solution of the Hamiltonian system* (2.1) *with*

$$A(t) = I - E^{-1}(t), \;\; B(t) = E^{-1}(t)F(t), \;\; C(t) = E^{-1}(t)G(t),$$

where $B(t)$ *and* $C(t)$ *are Hermitian and* $I - A(t)$ *is nonsingular.*

We now give another important example that leads to a Hamiltonian system. This example was given in [8, p. 85] (also, see [5, Equation (99)] and [7, Example 2]).

Example 2.5. The $2n$-th order scalar self–adjoint difference equation

$$L_{2n}y(t) = \sum_{i=0}^{n} \Delta^i[r_i(t)\Delta^i y(t-i)] = 0, \tag{2.4}$$

for $t \in [a+n, b+n]$, where the coefficient functions $r_i(t)$, $1 \leq i \leq n$, are real valued on $[a+n, b+n+i]$, respectively, and $r_n(t) \neq 0$ on $[a, b+2n]$, under the change of variables

$$x(t) = \begin{bmatrix} y(t-1) \\ \Delta y(t-1) \\ \cdots \\ \Delta^{n-1}y(t-n) \end{bmatrix}, \quad u(t) = \begin{bmatrix} (-1)^{n-1}\sum_{i=1}^{n} \Delta^{i-1}[r_i(t)\Delta^i y(t-i)] \\ \cdots \\ -\sum_{i=n-1}^{n} \Delta^{i-n+1}[r_i(t)\Delta^i y(t-i)] \\ r_n(t)\Delta^n y(t-n) \end{bmatrix},$$

is equivalent to the symplectic system (2.2) where

$$E(t) = \begin{bmatrix} 1 & \cdots & \cdots & 1 \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \quad F(t) = \frac{1}{r_n(t)} \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \cdots & \vdots & \vdots \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$G(t) = \begin{bmatrix} (-1)^n r_0(t) & (-1)^n r_0(t) & \cdots & (-1)^n r_0(t) \\ 0 & (-1)^{n-1} r_1(t) & \vdots & (-1)^{n-1} r_1(t) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -r_{n-1}(t) \end{bmatrix},$$

$$H(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & (-1)^n \frac{r_0(t)}{r_n(t)} \\ -1 & 1 & \ddots & \ddots & (-1)^{n-1} \frac{r_1(t)}{r_n(t)} \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & \left[1 - \frac{r_{n-1}(t)}{r_n(t)}\right] \end{bmatrix}$$

for $t \in [a+n, b+n]$. Since $E(t)$ is nonsingular we have by Theorem 2.4, that x, u is a solution of a Hamiltonian system.

An important concept for equation (2.4) is a generalized zero of order n which we now define.

Definition 2.6. We say that a nontrivial solution y of (2.4) has a generalized zero of order n at a if $y(a+i) = 0$, $0 \leq i \leq n-1$, and y has a generalized zero of order n at $t_0 > a$ provided $y(t_0 - 1) \neq 0$, $y(t_0 + i) = 0$, $0 \leq i \leq n-2$, and

$$(-1)^n r_n(t_0 + n - 1)y(t_0 - 1)y(t_0 + n - 1) \geq 0.$$

We say that the $2n$-th order self–adjoint difference equation $L_{2n}y(t) = 0$ is (n,n)-disconjugate on $[a+n, b+n]$ provided no nontrivial solution of $L_{2n}y(t) = 0$ has two or more distinct generalized zeros of order n.

See [8, pp. 168–183] for a discussion of equation (2.4). The following result is Corollary 1 in [7].

Theorem 2.7. *If*

$$(-1)^{n+i} r_i(t) \geq 0 \text{ on } [a+n, b+n+i] \quad \text{for} \quad 0 \leq i \leq n-1,$$

then $L_{2n}y(t) = 0$ *is* (n,n)*-disconjugate on* $[a, b+2n]$.

We now give another important example that leads to a Hamiltonian system. This example with a different change of variables was given in [8, p. 80].

Example 2.8. The self–adjoint vector difference equation

$$\Delta[P(t)\Delta y(t-1)] + Q(t)y(t) = 0, \tag{2.5}$$

for $t \in [a+1, b+1]$, $y(t) \in \mathbb{C}^n$, where $P(t)$ and $Q(t)$ are assumed to be $n \times n$ Hermitian matrix functions on $[a+1, b+2]$ and $[a+1, b+1]$, respectively and $P(t)$ is nonsingular on $[a+1, b+2]$ can be written as a Hamiltonian system

$$\Delta x(t) = 0 \cdot x(t+1) + P^{-1}(t)u(t), \quad \Delta u(t) = -Q(t)x(t+1) - 0 \cdot u(t).$$

Many papers have been devoted to the study of equation (2.5), e.g., [4, 43, 45, 46, 47].

If y is a solution of (2.5), then

$$y^*(t-1)P(t)y(t) - y^*(t)P(t)y(t-1) \quad \text{is constant}$$

for $t \in [a+1, b+2]$. If this constant is zero, then we say y is a *prepared* solution of (2.5). Hence a solution of (2.5) is prepared if and only if $y^*(t-1)P(t)y(t)$ is real on $[a+1, b+1]$.

We now define a *generalized zero* of a nontrivial prepared solution of (2.5).

Definition 2.9. Assume y is a nontrivial prepared solution of (2.5). We say that y has a generalized zero at a if $y(a) = 0$ and a generalized zero at $t_0 > a$ provided $y(t_0) \neq 0$ and

$$y^*(t_0 - 1)P(t_0)y(t_0) \leq 0.$$

The self–adjoint vector equation (2.5) is said to be *disconjugate* on $[a, b+2]$ provided no nontrivial prepared solution has two generalized zeros in $[a, b+2]$.

Peterson and Ridenhour prove the following interesting theorem in [45].

Theorem 2.10. *Assume $P(t) > 0$ (i.e., $P(t)$ is positive definite) and there is a scalar function $q(t)$ with $q(t) \geq 0$ and $Q(t) \leq q(t)I$ on $[a+1, b+1]$. Let*

$$D := 4\left\{\sum_{t+a}^{b+1} P^{-1}(t)\right\}^{-1} - \sum_{t=a+1}^{b+1} q(t)I.$$

If $D > 0$, then (2.5) is disconjugate on $[a, b+2]$.

3. SYMPLECTIC SYSTEMS

Let us have a closer look at an arbitrary *symplectic system* of the form (2.2). For convenience we introduce

$$z = \begin{pmatrix} x \\ u \end{pmatrix} \quad \text{and} \quad \mathcal{S} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

so that (2.2) may be rewritten as

$$z(t+1) = \mathcal{S}(t)z(t). \tag{3.1}$$

The assumption of symplecticity of (3.1) means that $\mathcal{S}^*(t)\mathcal{J}\mathcal{S}(t) = \mathcal{J}$ for all $t \in \mathbb{Z}$ (see Section 2). The matrices $\mathcal{S}(t)$ are of dimension $2n \times 2n$. We denote $2n \times n$ matrix-valued solutions of (3.1) by Z (in contrast to $\mathbb{R}^{2n}$-vector-valued solutions z). For two such solutions Z_1 and Z_2 we have

$$\begin{aligned} \Delta\left[Z_1^*(t)\mathcal{J}Z_2(t)\right] &= Z_1^*(t+1)\mathcal{J}Z_2(t+1) - Z_1^*(t)\mathcal{J}Z_2(t) \\ &= Z_1^*(t)\mathcal{S}^*(t)\mathcal{J}\mathcal{S}(t)Z_2(t) - Z_1^*(t)\mathcal{J}Z_2(t) \\ &= Z_1^*(t)\mathcal{J}Z_2(t) - Z_1^*(t)\mathcal{J}Z_2(t) = 0, \end{aligned}$$

which immediately yields the following easy but fundamental result.

Lemma 3.1 (Wronskian Identity)**.** *For two $2n \times n$ matrix-valued solutions Z_1 and Z_2 of (3.1), the* Wronskian *$Z_1^*(t)\mathcal{J}Z_2(t)$ is independent of $t \in \mathbb{Z}$.*

Also, in order to completely understand our next definition, observe that rank $Z(t)$ is independent of $t \in \mathbb{Z}$ for any solution Z of (3.1).

Definition 3.2. If $Z^*(t)\mathcal{J}Z(t)$ is always the zero matrix for a solution Z of (3.1), then we call Z a *conjoined* (*prepared*) *solution* of (3.1). If in addition rank $Z(t)$ is always n, Z is said to be a *conjoined basis* of (3.1). Finally, two conjoined bases Z and $\tilde{Z}$ are called *normalized* provided $Z^*(t)\mathcal{J}\tilde{Z}(t)$ is always the identity matrix.

Given a conjoined basis Z of (3.1), it is always possible to find another conjoined basis $\tilde{Z}$ such that Z and $\tilde{Z}$ are normalized conjoined bases. For this purpose one may take the unique solution $\tilde{Z}$ of the initial value problem

$$\tilde{Z}(t+1) = \mathcal{S}(t)\tilde{Z}(t), \quad \tilde{Z}(0) = \mathcal{J}^{-1}Z(0)\left[Z^*(0)Z(0)\right]^{-1},$$

and then

$$\tilde{Z}^*(t)\mathcal{J}\tilde{Z}(t) \equiv [Z^*(0)Z(0)]^{-1} Z^*(0)\mathcal{J}^{-1}\mathcal{J}\mathcal{J}^{-1}Z(0)\,[Z^*(0)Z(0)]^{-1} = 0$$

and

$$Z^*(t)\mathcal{J}\tilde{Z}(t) \equiv Z^*(0)\mathcal{J}\mathcal{J}^{-1}Z(0)\,[Z^*(0)Z(0)]^{-1} = I.$$

Next, let $t_0 \in \mathbb{Z}$ and consider the unique solution Z of

$$Z(t+1) = \mathcal{S}(t)Z(t), \quad Z(t_0) = \begin{pmatrix} 0 \\ I \end{pmatrix} \tag{3.2}$$

as well as the unique solution $\tilde{Z}$ of

$$Z(t+1) = \mathcal{S}(t)Z(t), \quad Z(t_0) = \begin{pmatrix} -I \\ 0 \end{pmatrix}.$$

We have $\operatorname{rank} Z(t) \equiv \operatorname{rank} \tilde{Z}(t) \equiv n$,

$$Z^*(t)\mathcal{J}Z(t) = (0 \quad I)\,\mathcal{J}\begin{pmatrix} 0 \\ I \end{pmatrix} = 0, \quad \tilde{Z}^*(t)\mathcal{J}\tilde{Z}(t) = (-I \quad 0)\,\mathcal{J}\begin{pmatrix} -I \\ 0 \end{pmatrix} = 0,$$

and

$$Z^*(t)\mathcal{J}\tilde{Z}(t) = (0 \quad I)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} -I \\ 0 \end{pmatrix} = (-I \quad 0)\begin{pmatrix} -I \\ 0 \end{pmatrix} = I.$$

Hence Z and $\tilde{Z}$ are (the *special*) normalized conjoined bases of (3.1) (at t_0). This Z is called the *principal solution* of (3.1) (at t_0) while $\tilde{Z}$ is referred to as being the *associated solution* of (3.1) (at t_0). A generalization of Definition 2.9 is now provided by the following (see also [18, Definitions 3 and 4]). We use the notation $\operatorname{Ker}M$, $\operatorname{Im}M$, and $M^\dagger$ to denote the kernel, the image, and the Moore–Penrose inverse of M, respectively.

Definition 3.3. A (vector-valued) solution z of (3.1) is said to have a *generalized zero* in $(t, t+1]$ if

$$x(t) \neq 0, \quad x(t+1) \in \operatorname{Im}F(t), \text{ and } x^*(t)F^\dagger(t)x(t+1) \leq 0.$$

System (3.1) is called *disconjugate* on $[a, b]$ if no solution of (3.1) has more than one generalized zero in $[a, b]$ and if no solution z of (3.1) with $x(a) = 0$ has any generalized zeros in $[a, b]$. Finally, a (matrix-valued) solution Z of (3.1) is said to have a *focal point* in $(t, t+1]$ if the conditions

$$\operatorname{Ker}X(t+1) \subset \operatorname{Ker}X(t) \quad \text{and} \quad X(t)X^\dagger(t+1)F(t) \geq 0$$

do not hold simultaneously.

The following so–called *Reid Roundabout Theorem* states some of the characterizations of disconjugacy (for a complete list see [18, Theorem 1] and [20]).

Theorem 3.4 (Reid Roundabout Theorem). *The following are equivalent:*

(i) *The system* (3.1) *is disconjugate on* $[a, b]$*;*

(ii) *the principal solution (at a) of* (3.1) *has no focal points in* $(a, b]$;

(iii) *the quadratic functional* $\mathcal{F}(z) = \sum_{t=a}^{b-1} z^*(t)\left\{\mathcal{S}^*(t)\mathcal{K}\mathcal{S}(t) - \mathcal{K}\right\} z(t)$ *is positive for all z with $\mathcal{K}z \neq 0$ and $\mathcal{K}z(a) = \mathcal{K}z(b) = 0$, where* $\mathcal{K} = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}$.

We note that condition (ii) is a discrete version of the strengthened Jacobi Condition (see also Section 6 below). Condition (ii) is the most practical condition among the given ones, in fact it is easy to calculate the principal solution of (3.1) as the solution to the initial value problem (3.2). Hence condition (ii) is useful to check disconjugacy (i) of (3.1) and also to check the so–called *positive definiteness* (iii) of $\mathcal{F}$ (we refer to Section 6 where it will be necessary to check such positive definiteness). For further results on transformations and reciprocity of symplectic systems we refer to [17, 18, 23]. Here we now turn our attention to linear Hamiltonian difference systems of the form (2.1), where the

$$B(t) \text{ are assumed to be invertible matrices for all } t \in \mathbb{Z}. \tag{3.3}$$

For the principal solution (X, U) (at a) of such a system we have the following:

$$X(a+1) = [I - A(a)]^{-1} X(a) + [I - A(a)]^{-1} B(a)U(a) = [I - A(a)]^{-1} B(a).$$

Hence condition (ii) from Theorem 3.4 is satisfied provided

$$\left.\begin{array}{c} X(t+1) \text{ is invertible and} \\ X(t)X^{-1}(t+1)\,[I - A(t)]^{-1} B(t) > 0 \end{array}\right\} \text{ for all } a \le t \le b-1. \tag{3.4}$$

Let us assume (3.4) and put $Q(t) = U(t)X^{-1}(t)$ for $a < t \le b$. Then

$$I + B(t)Q(t) = [X(t) + B(t)U(t)]\, X^{-1}(t) = [I - A(t)]\, X(t+1)X^{-1}(t)$$

is invertible for $a < t < b$ and

$$X(t)X^{-1}(t+1)\,[I - A(t)]\, B(t) = [I + B(t)Q(t)]^{-1} B(t) > 0 \quad \text{for all} \quad a < t < b.$$

Also, Q is Hermitian, due to $U^*X = X^*U$, and

$$\begin{aligned} & C(t) + [I - A^*(t)]\, Q(t)\, [I + B(t)Q(t)]^{-1} [I - A(t)] \\ = \;& C(t) + [I - A^*(t)]\, U(t)X^{-1}(t+1) \\ = \;& U(t+1)X^{-1}(t+1) = Q(t+1). \end{aligned}$$

Moreover, we have the following fundamental relationship between linear Hamiltonian difference systems (2.1) and so–called *Riccati matrix difference equations* (for applications see the book [8] and the references therein)

$$Q(t+1) = C(t) + [I - A^*(t)]\, Q(t)\, [I + B(t)Q(t)]^{-1} [I - A(t)]. \tag{3.5}$$

Theorem 3.5. *Assume* (3.3). *Then* (2.1) *is disconjugate on* $[a,b]$ *if and only if there exists a symmetric solution* Q *of* (3.5) *on* $[a,b]$ *with* $[I+B(t)Q(t)]^{-1}B(t)>0$ *for all* $a<t<b$.

4. DISCRETE TRIGONOMETRIC MATRIX FUNCTIONS

We now give a very interesting example of a symplectic system (hence a Hamiltonian system if a certain matrix is nonsingular) that was introduced by D. Anderson in [9, 10]. Assume Q is an $n\times n$ Hermitian matrix function on the discrete interval $[a,\infty)$, then we define the discrete sine and cosine matrix functions

$$S(t)=S(t;a,Q),\qquad C(t)=C(t;a,Q)$$

to be the unique solution of the initial value problem

$$\begin{cases} S(t+1)=(\cos Q(t))\,S(t)+(\sin Q(t))\,C(t),\\ C(t+1)=-\,(\sin Q(t))\,S(t)+(\cos Q(t))\,C(t),\end{cases}\tag{4.1}$$

$$S(a)=0,\ \ C(a)=I,\tag{4.2}$$

where $\sin Q(t)$ and $\cos Q(t)$ are defined by their respective Taylor expansions.

Example 4.1. If $Q(t)$ in the matrix system (4.1) is diagonal, then

$$S(t;a,Q)=\sin\left(\sum_{\tau=a}^{t-1}Q(\tau)\right)\quad\text{and}\quad C(t;a,Q)=\cos\left(\sum_{\tau=a}^{t-1}Q(\tau)\right).$$

Anderson proved several very interesting results concerning these trigonometric functions some of which we state now. All of these results are in [9, 10]. For some further results along this line see [11, 21].

Theorem 4.2. *For* $t\ge a$ *the discrete trigonometric functions* $S(t)$, $C(t)$ *satisfy the following:*

$$S^*(t)S(t)+C^*(t)C(t)=I,\quad S(t)S^*(t)+C(t)C^*(t)=I,$$
$$S^*(t)C(t)=C^*(t)S(t),\quad S(t)C^*(t)=C(t)S^*(t),$$

and $\|S(t)\|_2^2+\|C(t)\|_2^2=n$.

We let $S(t;s)$, $C(t;s)$ be the unique solution of (4.1), (4.2) with a replaced by s, for any s in the discrete interval $[a,\infty)$. Then we have the following results.

Theorem 4.3 (Difference and Addition Formulae). *If* $t,s\in[a,\infty)$, *then*

$$S(t;s)=S(t;a)C^*(s;a)-C(t;a)S^*(s;a),\quad C(t;s)=C(t;a)C^*(s;a)+S(t;a)S^*(s;a)$$

and

$$S(t;a)=S(t;s)C(s;a)+C(t;s)S(s;a),\quad C(t;a)=C(t;s)C(s;a)-S(t;s)S(s;a).$$

For those values of t for which $C^{-1}(t)$ exists, define the discrete tangent matrix function

$$T(t) := C^{-1}(t)S(t).$$

In the same way, for those values of t for which $S^{-1}(t)$ exists define the discrete cotangent matrix function

$$\mathrm{COT}(t) := S^{-1}(t)C(t).$$

Theorem 4.4. *For any $t \geq a$ such that $C(t)$ is nonsingular, we have*

$$T^*(t) = T(t) \quad \text{and} \quad I + T^2(t) = [C^*(t)C(t)]^{-1}.$$

Moreover, for any $t \geq a$ such that $S(t)$ is nonsingular, we have

$$\mathrm{COT}^*(t) = \mathrm{COT}(t) \quad \text{and} \quad \mathrm{COT}^2(t) + I = [S^*(t)S(t)]^{-1}.$$

Anderson goes on to prove many other formulae of this type, proves some separation theorems, and considers oscillation of these discrete trigonometric matrix functions.

Lemma 4.5. *If $C(t)$ is nonsingular for $t \in \{t_0, t_0+1\}$, then*

$$\Delta T(t_0) = C^{-1}(t_0+1)\,(\sin Q(t_0))\,C^{*-1}(t_0).$$

Similarly,

$$\Delta \mathrm{COT}(t_0) = -S^{-1}(t_0+1)\,(\sin Q(t_0))\,S^{*-1}(t_0)$$

provided $S(t)$ is nonsingular for $t \in \{t_0, t_0+1\}$.

In the remainder of this section we will look at general *discrete trigonometric systems*. By this we mean a symplectic system (3.1) where the matrices $\mathcal{S}(t)$ (besides being symplectic) satisfy the additional assumption

$$\mathcal{J}^*\mathcal{S}(t)\mathcal{J} = \mathcal{S}(t). \tag{4.3}$$

For the system (4.1), condition (4.3) is easily verified:

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} \cos Q & \sin Q \\ -\sin Q & \cos Q \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} \cos Q & \sin Q \\ -\sin Q & \cos Q \end{pmatrix}.$$

Suppose z is a solution of the trigonometric system (3.1). Then

$$\mathcal{J}z(t+1) = -\mathcal{J}^*\mathcal{S}(t)z(t) = -\mathcal{J}^*\mathcal{S}(t)\mathcal{J}\mathcal{J}^*z(t) = \mathcal{S}(t)\mathcal{J}z(t).$$

Hence we can make the following easy observation.

Lemma 4.6. *Assume* (4.3). *If z solves* (3.1), *then so does $\mathcal{J}z$.*

It is also easy to see that any trigonometric system can be written as

$$x(t+1) = \mathcal{P}(t)x(t) + \mathcal{Q}(t)u(t), \quad u(t+1) = -\mathcal{Q}(t)x(t) + \mathcal{P}(t)u(t), \tag{4.4}$$

where $\mathcal{P}^T\mathcal{P} + \mathcal{Q}^T\mathcal{Q} = I = \mathcal{P}\mathcal{P}^T + \mathcal{Q}\mathcal{Q}^T$ and both $\mathcal{P}^T\mathcal{Q}$ and $\mathcal{P}\mathcal{Q}^T$ are symmetric. The importance of trigonometric systems lies in the following fact: Every symplectic system (3.1) may be transformed into a trigonometric system (4.4) by using a transformation that preserves oscillatory behavior of the original system. To be more precise, we have the following result, which is Theorem 3 in [21].

Theorem 4.7. *Given a symplectic system* (3.1), *it is possible to find* $n \times n$*-matrix-valued functions* H *and* K *such that* H *is nonsingular and* H^TK *is symmetric, and the transformation*

$$\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} H^{-1} & 0 \\ -K^T & H^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

transforms (3.1) *into a trigonometric system* (4.4) *with*

$$\begin{cases} \mathcal{P}(t) = H^{-1}(t+1)E(t)H(t) + H^{-1}(t+1)F(t)K(t) \\ \qquad \mathcal{Q}(t) = H^{-1}(t+1)F(t)\left(H^T(t)\right)^{-1}. \end{cases}$$

More precisely, if $Z = \binom{X}{U}$ *and* $\tilde{Z} = \binom{\tilde{X}}{\tilde{U}}$ *are normalized conjoined bases of (S), then the matrices* H *and* K *are given by*

$$HH^T = XX^T + \tilde{X}\tilde{X}^T \quad \text{and} \quad K = \left(UX^T + \tilde{U}\tilde{X}^T\right)\left(H^T\right)^{-1}.$$

Furthermore, this transformation preserves oscillatory behavior.

5. DOMINANT AND RECESSIVE SOLUTIONS

In this section we introduce the following very important reduction of order formula for the symplectic system (2.2).

Theorem 5.1 (Reduction of Order Theorem)**.** *Assume* (X_0, U_0) *is a conjoined solution of the symplectic system* (2.2) *such that* $X_0(t)$ *is invertible on* $[a, b+1]$. *If* (X, U) *is a solution of* (2.2), *then*

$$X(t) = X_0(t)[P + S_0(t)]Q \quad \text{and} \quad U(t) = U_0(t)[P + S_0(t)Q] + X_0^{*-1}(t)Q \tag{5.1}$$

for $a \le t \le b+1$, *where*

$$S_0(t) = \sum_{s=a}^{t-1} X_0^{-1}(s+1)F(s)X_0^{*-1}(s),$$

$$P = X_0^{-1}(a)X(a), \quad \text{and} \quad Q = X_0^*(t)U(t) - U_0^*(t)X(t). \tag{5.2}$$

Conversely, if P *and* Q *are constant* $n \times n$*-matrices and* X, U *are defined by* (5.1), *then* (X, U) *is a solution of* (2.2) *and equations* (5.2) *hold. Furthermore* (X, U) *is a prepared solution if and only if* P^*Q *is Hermitian.*

For the proof of this reduction of order theorem see [8, p. 105]. The reduction of order formula for the three term second order matrix equation

$$-K(t)Y(t+1)+N(t)Y(t)-K^*(t-1)Y(t-1)=0$$

is given in Section 5.8 in [8].

The following interesting simple example of the above reduction of order theorem is given by Anderson in [10]. Note that the formulae in the example below for the discrete tangent matrix function and the discrete cotangent matrix function follow immediately from the reduction of order theorem.

Theorem 5.2. *If the cosine matrix function $C(t)$ is nonsingular on $[a,b]$, then*

$$C^{-1}(t+1)\,(\sin Q(t))\,C^{*-1}(t)$$

is Hermitian on $[a,b-1]$ and the tangent matrix function defined in Section 4 is given by

$$T(t)=\sum_{\tau=a}^{t-1} C^{-1}(\tau+1)\,(\sin Q(\tau))\,C^{*-1}(\tau)$$

on $[a,b]$. If the matrix sine function $S(t)$ is nonsingular on $[a+1,b]$, then

$$S^{-1}(t+1)\,(\sin Q(t))\,S^{*-1}(t)$$

is Hermitian on $[a+1,b-1]$, and the cotangent matrix function is given by

$$\mathrm{COT}(t)=-\sum_{\tau=a+1}^{t-1}\left[S^{-1}(\tau+1)\,(\sin Q(\tau))\,S^{*-1}(\tau)\right]+\mathrm{COT}(a+1)$$

for $t\in[a+2,b]$.

We motivate the discrete theory with the continuous example $x''-x=0$ which has general solution $x(t)=C_1e^t+C_2e^{-t}$. The solution e^t is called dominant at ∞ and the solution e^{-t} is called recessive at ∞. Note that if $x(t)$ is linearly independent of e^{-t}, then $e^{-t}/x(t)$ goes to 0 as $t\to\infty$. The discrete example of the Fibonacci recurrence $x(t+1)=x(t)+x(t-1)$ has general solution $x(t)=C_1r_1^t+C_2r_2^t$ where $r_1=(1+\sqrt{5})/2$ and $r_2=(1-\sqrt{5})/2$ are the roots of the characteristic equation $r^2=r+1$. Here the solution $x_1(t)=r_1^t$ is dominant at ∞ and $x_2(t)=r_2^t$ is recessive at ∞. Note that $x_2(t)/x(t)\to 0$ as $t\to\infty$ if the solution $x(t)$ is linearly independent of the recessive solution $x_2(t)$.

Equations (5.1) of the Reduction of Order Theorem, Theorem 5.1, allow us to express all solutions on $[a,\infty)$ in terms of one conjoined solution (X_0,U_0) with $X_0(t)$ invertible on $[a,\infty)$. Furthermore $S_0(t)$ is Hermitian (see [8, Theorem 3.33, pp. 109–110]).

A conjoined solution $(X_0(t), U_0(t))$ defined for large t is said to be *dominant at* ∞ if there exists an integer a such that $X_0(t)$ is nonsingular on $[a, \infty)$ and $S_0(t)$ converges as $t \to \infty$ to a (Hermitian) matrix with finite entries.

A conjoined basis $Z_0(t) = \binom{X_0(t)}{U_0(t)}$ is said to be *recessive at* ∞ if whenever $Z(t) = \binom{X(t)}{U(t)}$ is a $2n \times n$ solution such that the Wronskian $Z_0^*(t)\mathcal{J}Z(t)$ is nonsingular, then there exists an integer a such that $X(t)$ is nonsingular for $t \geq a$ and

$$\lim_{t\to\infty} X^{-1}(t)X_0(t) = 0.$$

The following is Theorem 3.43 of [8, p. 118].

Theorem 5.3. *Suppose $Z_0(t) = \binom{X_0(t)}{U_0(t)}$ is a solution which is recessive at ∞.*

(i) *If K is a nonsingular $n \times n$ constant matrix, then the solution $(X_0(t)K, U_0(t)K)$ is recessive at ∞.*
(ii) *If $Z(t) = \binom{X(t)}{U(t)}$ is a conjoined solution such that the Wronskian $Z_0^*(t)\mathcal{J}Z(t)$ is nonsingular, then $(X(t), U(t))$ is dominant at ∞.*

Theorem 5.4 (Connection Theorem)**.** *Let $Z(t) = \binom{X(t)}{U(t)}$ be the principal solution at a and let $\tilde{Z}(t) = \binom{\tilde{X}(t)}{\tilde{U}(t)}$ be the associated solution at a. Then the following conditions are equivalent:*

(i) *Z is dominant at ∞.*
(ii) *$X(t)$ is invertible for sufficiently large t and there exists a Hermitian $n\times n$ matrix Ω with finite entries such that*

$$\lim_{t\to\infty} X^{-1}(t)\tilde{X}(t) = -\Omega.$$

(iii) *There exists a solution $Z_0(t) = \binom{X_0(t)}{U_0(t)}$ which is recessive at ∞ and has $X_0(a)$ nonsingular.*

Furthermore, if (i),(ii), or (iii) holds, then any solution $Z_0(t) = \binom{X_0(t)}{U_0(t)}$ which is recessive at ∞ has $X_0(a)$ nonsingular and $U_0(a)X_0^{-1}(a) = -\Omega$.

For the proof of this theorem, see §3.13 of [8, pp. 125–130]. This "Connection Theorem" connects the existence of a recessive solution with $X_0(a)$ nonsingular to the convergence to the matrix Ω of an associated matrix continued fraction. An introductory treatment of continued fractions is presented in Chapter 2 of [8]. Furthermore, the recessive solution corresponds with the eventually minimal solution of the associated matrix Riccati equation. See [4] and §2.6 of [8] for discussions of the continued fraction representation of this minimal solution.

As an additional consequence of any one of conditions (i), (ii), or (iii) of the above theorem, we have the following asymptotic result (Theorem 3.53 of [8, p. 132]): If $(X_1(t), U_1(t))$ is any solution, then

$$\lim_{t\to\infty} X^{-1}(t)X_1(t) = \Omega X_1(a) + U_1(a).$$

The Olver–Reid construction (Theorem 3.56 of [8, p. 134]) gives a construction of the recessive solution $(X_0(t), U_0(t))$ with $X_0(a) = I$ by means of a limiting two point boundary value problem.

6. DISCRETE CALCULUS OF VARIATIONS

In this section we are mainly concerned with a discrete version of the classical problem of the calculus of variations, namely with the following *variational problem*:

$$(6.1)\qquad \begin{cases} \text{Minimize the functional} \\ \mathcal{F}(x) = \sum_{t=a}^{b-1} f(t, x(t+1), \Delta x(t)) \text{ for all admissible } x, \text{ i.e.,} \\ (x(t+1), \Delta x(t)) \in G \text{ for all } a \le t \le b-1, \\ x : [a,b] \cap \mathbb{Z} \to \mathbb{R}^n,\ x(a) = \alpha,\ x(b) = \beta, \end{cases}$$

where $n \in \mathbb{N}$, $a, b \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{R}^n$, $G \subset \mathbb{R}^{2n}$ is a domain, and $f : ([a,b] \cap \mathbb{Z}) \times G \to \mathbb{R}$ is a function with "$f \in \mathrm{C}^2(G)$", i.e., f is of class C^2 with respect to the components of the second and third variables.

Definition 6.1. An admissible (sequence) $\hat{x}$ is called a local minimum of (6.1) if there exists $\varepsilon > 0$ such that

$$\|x - \hat{x}\| = \min\{|x(t) - \hat{x}(t)| :\ a \le t \le b\} < \varepsilon, \quad x \quad \text{admissible}$$

implies $\mathcal{F}(\hat{x}) \le \mathcal{F}(x)$.

If $\hat{x}$ is a local minimum of (6.1), then it is clear that the function $\phi_\eta(\varepsilon) = \mathcal{F}(\hat{x}+\varepsilon\eta)$ has a local minimum at 0 whenever η is a so–called *admissible variation*, namely a function $\eta : [a,b] \cap \mathbb{Z} \to \mathbb{R}$ with $\eta(a) = \eta(b) = 0$. Hence $\phi_\eta'(0) = 0$ and $\phi_\eta''(0) \ge 0$ for all admissible variations η. We define the first and second discrete (Gâteaux) variations of $\mathcal{F}$ at $\hat{x}$ by $\mathcal{F}_1(\eta) = \phi_\eta'(0)$ and $\mathcal{F}_2(\eta) = \phi_\eta''(0)$, respectively, and compute

$$\mathcal{F}_1(\eta) = \sum_{t=a}^{b-1} \left\{\eta^T(t+1) f_x\left(t, \hat{x}(t+1), \Delta\hat{x}(t)\right) + (\Delta\eta(t))^T f_v\left(t, \hat{x}(t+1), \Delta\hat{x}(t)\right)\right\}$$

and

$$\begin{aligned} \mathcal{F}_2(\eta) \; = \; \sum_{t=a}^{b-1} & \left\{\eta^T(t+1) f_{xx}\left(t, \hat{x}(t+1), \Delta\hat{x}(t)\right)\eta(t+1)\right. \\ & +2\eta^T(t+1) f_{xv}\left(t, \hat{x}(t+1), \Delta\hat{x}(t)\right)\Delta\eta(t) \\ & \left.+(\Delta\eta(t))^T f_{vv}\left(t, \hat{x}(t+1), \Delta\hat{x}(t)\right)\Delta\eta(t)\right\}, \end{aligned}$$

where we have used f_x and f_v as the gradients of $f(t,x,v)$ with respect to x and v, respectively (column vectors), and f_{xx}, f_{xv}, f_{vv} as the corresponding Hessian matrices. Note that f_{xx} and f_{vv} are symmetric $n \times n$ matrices due to our general assumptions.

For an admissible variation η, we may rewrite $\mathcal{F}_1(\eta)$ using the discrete product rule as

$$\begin{aligned}\mathcal{F}_1(\eta) &= \sum_{t=a}^{b-1}\{\eta^T(t+1)f_x(t,\hat{x}(t+1),\Delta\hat{x}(t)) + \Delta[\eta^T(t)f_v(t,\hat{x}(t+1),\Delta\hat{x}(t))] \\ &\qquad -\eta^T(t+1)\Delta f_v(t,\hat{x}(t+1),\Delta\hat{x}(t))\} \\ &= \sum_{t=a}^{b-1}\eta^T(t+1)\{f_x(t,\hat{x}(t+1),\Delta\hat{x}(t)) - \Delta f_v(t,\hat{x}(t+1),\Delta\hat{x}(t))\}.\end{aligned}$$

Now, by the (easy to prove) discrete version of the Fundamental Lemma of the calculus of variations, we have that $\mathcal{F}_1(\eta)=0$ for all admissible variations η if and only if the discrete *Euler–Lagrange equation*

$$f_x(t,\hat{x}(t+1),\Delta\hat{x}(t)) - \Delta f_v(t,\hat{x}(t+1),\Delta\hat{x}(t)) = 0 \quad \text{for all} \quad a\le t\le b-2 \tag{6.2}$$

is satisfied. We need one more piece of terminology.

Definition 6.2. The functional $\mathcal{F}_2$ is called *positive semidefinite* provided $\mathcal{F}_2(\eta)\ge 0$ for all admissible variations η. Also, $\mathcal{F}_2$ is called *positive definite* if $\mathcal{F}_2(\eta)>0$ whenever η is a nontrivial admissible variation.

The following result is now obvious.

Theorem 6.3 (Necessary Conditions)**.** *Let $\hat{x}$ be a local minimum of* (6.1). *Then the Euler–Lagrange equation* (6.2) *holds and the second variation of $\mathcal{F}$ at $\hat{x}$ is positive semidefinite.*

Now, any solution of (6.2) is called an *extremal.* Of course we are interested in extracting the solutions of (6.1) from the set of the extremals. For an extremal $\hat{x}$ we introduce the notation

$$\begin{cases} Q(t) = f_{xx}(t,\hat{x}(t+1),\Delta\hat{x}(t)), \\ R(t) = f_{xv}(t,\hat{x}(t+1),\Delta\hat{x}(t)), \\ P(t) = f_{vv}(t,\hat{x}(t+1),\Delta\hat{x}(t)). \end{cases}$$

By considering admissible variations of the form $\eta(t)=\gamma\in\mathbb{R}^n$ for $t=m$ and 0 otherwise (where $a+1\le m\le b-1$), Theorem 6.3 immediately yields the discrete version of Legendre's condition which surprisingly (considering the analogy of the Euler–Lagrange equation etc. to the "continuous" case) differs from the classical version significantly.

Theorem 6.4 (Legendre's Necessary Condition)**.** *Let $\hat{x}$ be a local minimum of* (6.1). *Then the Legendre condition*

$$P(t)+R(t)+R^T(t)+Q(t)+P(t+1)\ge 0 \quad \textit{for all} \quad a\le t\le b-2 \tag{6.3}$$

is satisfied.

Remark 6.5 (The Legendre Condition). Here we want to make an attempt to compare the discrete Legendre condition (6.3) to the continuous Legendre condition

$$P(t) = f_{vv}\left(t, \hat{x}(t+1), \Delta\hat{x}(t)\right) \geq 0 \quad \text{for all} \quad a \leq t \leq b \tag{6.4}$$

which is necessary to hold if $\hat{x}$ is a local minimum of

$$\mathcal{F}(x) = \int_a^b f(t, x(t), \dot{x}(t))\mathrm{d}t \to \min, \quad x(a) = \alpha,\ x(b) = \beta \tag{6.5}$$

(see e.g. [41, Theorem 8.2.3] for more details). For this, we look at a problem of the form

$$\begin{cases} \mathcal{F}^{(N)}(x) = \sum_{t\in\mathcal{T}^{(N)}} f\left(t, x\left(t+\frac{1}{N}\right), N\left[x\left(t+\frac{1}{N}\right) - x(t)\right]\right) \to \min, \\ x(a) = \alpha, x(b) = \beta, \end{cases} \tag{6.6}$$

where $a, b \in \mathbb{Z}$, $N \in \mathbb{N}$, and $\mathcal{T}^{(N)} = \left\{\frac{k}{N} \middle| \, k \in \mathbb{Z}\right\} \cap \left[a, b - \frac{1}{N}\right]$. The problem (6.6) may be viewed as a discretization of (6.5), and, furthermore, it is the same as our original problem (6.1) in case $N = 1$. Local minima and admissible variations for (6.6) are defined accordingly. As before, if $\hat{x}$ is a local minimum of (6.6), then $\phi_\eta^{(N)}(\varepsilon) = \mathcal{F}^{(N)}(\hat{x}+\varepsilon\eta)$ has a local minimum at 0 whenever η is an admissible variation. This implies that

$$\begin{aligned} \mathcal{F}_2^{(N)}(\eta) = \sum_{t\in\mathcal{T}^{(N)}} \Bigg\{ & \eta^T\left(t+\frac{1}{N}\right) Q(t)\eta\left(t+\frac{1}{N}\right) \\ & +2N\eta^T\left(t+\frac{1}{N}\right) R(t)\left[\eta\left(t+\frac{1}{N}\right) - \eta(t)\right] \\ & +N^2\left[\eta\left(t+\frac{1}{N}\right) - \eta(t)\right]^T P(t)\left[\eta\left(t+\frac{1}{N}\right) - \eta(t)\right]\Bigg\} \end{aligned}$$

is nonnegative for any admissible variation η, where we put

$$\begin{cases} Q(t) = f_{xx}(\cdot), R(t) = f_{xv}(\cdot), P(t) = f_{vv}(\cdot) \\ \text{with} \quad (\cdot) = \left(t, \hat{x}\left(t+\frac{1}{N}\right), N\left[\hat{x}\left(t+\frac{1}{N}\right) - \hat{x}(t)\right]\right). \end{cases}$$

Hence, by considering admissible variations of the form $\eta(t) = \frac{\gamma}{N} \in \mathbb{R}^n$ for $t = m$ and 0 otherwise (where $m \in \mathcal{T}^{(N)}$ with $a + \frac{1}{N} \leq m \leq b - \frac{1}{N}$), we obtain the Legendre necessary condition for (6.6) as

$$P(t) + \frac{1}{N}R(t) + \frac{1}{N}R^T(t) + \frac{1}{N^2}Q(t) + P\left(t+\frac{1}{N}\right) \geq 0 \quad \text{for all} \quad t \in \mathcal{T}^{(N)} \setminus \left\{b - \frac{1}{N}\right\}.$$

For $N = 1$ this is of course our discrete Legendre condition (6.3). However, as $N \to \infty$ (observe that P, Q, and R also depend on N), we obtain the continuous Legendre condition (6.4) in the limit. We here finish our investigation of problem (6.6) and return to our original object of interest (6.1).

Now assume the *discrete strengthened Legendre condition*

$$P(t) \text{ and } P(t)+R(t) \quad \text{are invertible for all} \quad a \le t \le b-1 \tag{6.7}$$

and define A, B, and C by

$$A = -P^{-1}R^T, \quad B = P^{-1}, \quad C = Q - RP^{-1}R^T. \tag{6.8}$$

Then we may rewrite $\mathcal{F}_2$ as follows:

$$\begin{aligned}
\mathcal{F}_2(\eta) &= \sum_{t=a}^{b-1} \Big\{ \eta^T(t+1)Q(t)\eta(t+1) + 2\eta^T(t+1)R(t)\Delta\eta(t) \\
&\qquad + (\Delta\eta(t))^T P(t)\Delta\eta(t) \Big\} \\
&= \sum_{t=a}^{b-1} \Big\{ \eta^T(t+1)C(t)\eta(t+1) + \eta^T(t+1)R(t)P^{-1}(t)R^T(t)\eta(t+1) \\
&\qquad + 2\eta^T(t+1)R(t)P^{-1}(t)P(t)\Delta\eta(t) + (P(t)\Delta\eta(t))^T P^{-1}(t)P(t)\Delta\eta(t) \Big\} \\
&= \sum_{t=a}^{b-1} \Big\{ \eta^T(t+1)C(t)\eta(t+1) \\
&\qquad + \left[P(t)\Delta\eta(t) + R^T(t)\eta(t+1)\right]^T B(t) \left[P(t)\Delta\eta(t) + R^T(t)\eta(t+1)\right] \Big\} \\
&= \sum_{t=a}^{b-1} \left\{ \eta^T(t+1)C(t)\eta(t+1) + \xi^T(t)B(t)\xi(t) \right\}
\end{aligned}$$

with $\xi(t) = P(t)\Delta\eta(t) + R^T(t)\eta(t+1)$, i.e., $\Delta\eta(t) = A(t)\eta(t+1) + B(t)\xi(t)$. The question of positive definiteness of the *discrete quadratic functional* $\mathcal{F}_2$ is now satisfactorily answered by the following theorem on the necessity of the strengthened Jacobi condition.

Theorem 6.6. *Let the strengthened Legendre condition* (6.7) *hold. Then* $\mathcal{F}_2$ *is positive definite if and only if the principal solution of the linear Hamiltonian difference system*

$$\Delta\eta(t) = A(t)\eta(t+1) + B(t)\xi(t), \quad \Delta\xi(t) = C(t)\eta(t+1) - A^T(t)\xi(t) \tag{6.9}$$

(*with notation* (6.8)) *has no focal points in* $(a, b]$.

We refer to Theorem 3.4 for a more general version of the above Theorem 6.6. Here we note that (6.9) is indeed a linear Hamiltonian difference system since both B and C are symmetric and since $I - A$ is invertible because of the strengthened Legendre condition (6.7).

Remark 6.7 (The Strengthened Legendre Condition)**.** Although we tried to motivate our notion of the discrete strengthened Legendre condition in Remark 6.5, it is not completely satisfactory, as e.g. it does not directly imply the discrete Legendre condition. Such an implication of course is true in the continuous case, where the strengthened Legendre condition reads $P(t) > 0$ for all $t \in [a, b]$ (see e.g. [41, Lemma 8.2.4]). Actually, in contrast to the continuous case, another form of the discrete strengthened Legendre condition may be used that still yields a statement in the spirit of Theorem 6.6, namely

$$R(t) \text{ symmetric and } P(t) + R(t) \text{ invertible for all } a \leq t \leq b-1. \tag{6.10}$$

(Note that, in general, $R(t)$ is not symmetric. However, if $R(t)$ is symmetric (e.g., if $n = 1$), then (6.10) is an assumption that is more often satisfied than (6.7).) To support our statement we introduce

$$\tilde{P} = P + R \quad \text{and} \quad \tilde{Q} = Q - \Delta R$$

so that $\mathcal{F}_2$ may be rewritten as follows:

$$\begin{aligned}
\mathcal{F}_2(\eta) &= \sum_{t=a}^{b-1} \{\eta^T(t+1)Q(t)\eta(t+1) + 2\eta^T(t+1)R(t)\Delta\eta(t) \\
&\qquad + (\Delta\eta(t))^T P(t)\Delta\eta(t)\Big\} \\
&= \sum_{t=a}^{b-1} \{\eta^T(t+1)Q(t)\eta(t+1) + \eta^T(t+1)R(t)\eta(t+1) - \eta^T(t)R(t)\eta(t) \\
&\qquad + (\Delta\eta(t))^T R(t)\Delta\eta(t) + (\Delta\eta(t))^T P(t)\Delta\eta(t)\Big\} \\
&= \sum_{t=a}^{b-1} \Big\{\eta^T(t+1)\tilde{Q}(t)\eta(t+1) + (\Delta\eta(t))^T \tilde{P}(t)\Delta\eta(t)\Big\}.
\end{aligned}$$

This form of $\mathcal{F}_2$ looks just like the previous one, except that the mixed term vanishes now, i.e., $\tilde{R}(t) = 0$. With this notation the Legendre condition (6.3) turns out to be

$$\begin{aligned}
0 &\leq \tilde{P}(t) + \tilde{Q}(t) + \tilde{P}(t+1) \\
&= P(t) + R(t) + Q(t) - R(t+1) + R(t) + P(t+1) + R(t+1) \\
&= P(t) + 2R(t) + Q(t) + P(t+1)
\end{aligned}$$

which is just the same as (6.3). However, the strengthened Legendre condition (6.7) "simplifies" to $\tilde{P}(t)$ invertible, i.e., to (6.10). To conclude this remark, let us point out some differences from the continuous case. Surely the assumption on the invertibility of $I - A$ is not needed in the continuous case, and $P > 0$ is to be used for the strengthened Legendre condition. However, if one attempts to find $\tilde{P}$ and $\tilde{Q}$ as in the previous discussion to remove the mixed term, then one will be led to (provided $R(t)$

is symmetric and differentiable)

$$\begin{aligned}\mathcal{F}_2(\eta) &= \int_a^b \{\eta^T Q\eta + 2\eta^T R\dot{\eta} + \dot{\eta}^T P\dot{\eta}\}(t)\mathrm{d}t \\ &= \int_a^b \Big\{\eta^T(t)Q(t)\eta(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\eta^T(t)R(t)\eta(t)\right] \\ &\qquad -\eta^T(t)\dot{R}(t)\eta(t) + \dot{\eta}^T(t)P(t)\dot{\eta}(t)\Big\}\mathrm{d}t \\ &= \int_a^b \Big\{\eta^T\tilde{Q}\eta + \dot{\eta}^T\tilde{P}\dot{\eta}\Big\}(t)\mathrm{d}t\end{aligned}$$

with $\tilde{Q} = Q - R'$ and $\tilde{P} = P$, and hence one ends up just as before by demanding that $\tilde{P} = P > 0$.

In view of the above Remark 6.7 we have the following result.

Theorem 6.8. *Assume that the* (*alternative*) *strengthened Legendre condition* (6.10) *holds. Then $\mathcal{F}_2$ is positive definite if and only if the principal solution of the linear Hamiltonian difference system*

$$\Delta\eta(t) = \tilde{P}^{-1}(t)\xi(t), \quad \Delta\xi(t) = \tilde{Q}(t)\eta(t+1) \tag{6.11}$$

has no focal points in $(a, b]$.

To provide a deeper understanding of the connection of system (6.9) with $\mathcal{F}_2$, we observe that (if $\hat{x}$ is a local minimum of (6.1)) by Theorem 6.3

$$\mathcal{F}_2(\eta) = \sum_{t=a}^{b-1} \Omega(t, \eta(t+1), \Delta\eta(t)) \geq 0 = \mathcal{F}_2(0)$$

whenever $\eta(a) = \eta(b) = 0$, where we put

$$\Omega(t, \eta, \zeta) = \eta^T Q(t) + 2\eta^T R(t)\zeta + \zeta^T P(t)\zeta.$$

Hence 0 is a solution of a special problem of the form (6.1), the so–called *discrete accessory minimum problem*

$$\mathcal{F}_2(\eta) = \sum_{t=a}^{b-1} \Omega(t, \eta(t+1), \Delta\eta(t)) \to \min, \quad \eta(a) = \eta(b) = 0.$$

Now, whenever $\hat{\eta}$ is a local minimum of the problem (6.12), then by Theorem 6.3 the Euler–Lagrange equation

$$Q(t)\hat{\eta}(t+1) + R(t)\Delta\hat{\eta}(t) = \Delta\left[P(t)\Delta\hat{\eta}(t) + R^T(t)\eta(t+1)\right] \tag{6.12}$$

must be satisfied for all $a \leq t \leq b-2$. We call (6.12) the *Jacobi difference equation.* If $R = 0$, then the Jacobi difference equation reduces to a self–adjoint vector difference equation or (in the scalar case) to a Sturm–Liouville difference equation of second order. Observe also the connection of (6.12) to (6.9) and (6.11).

7. DIFFERENTIAL EQUATIONS ON TIME SCALES

In this section we would like to discuss some results for Hamiltonian systems on a time scale (or *measure chain*) $\mathbb{T}$. Many of the results discussed here will be for the self–adjoint differential equation

$$Lx(t) := [p(t)x^{\Delta}]^{\Delta}(t) + q(t)x^{\sigma}(t) = 0. \tag{7.1}$$

To understand this so-called differential (or dynamic) equation on a time scale $\mathbb{T}$ we need some preliminary definitions. We assume throughout that $\mathbb{T}$ is a closed subset of the real numbers $\mathbb{R}$ that is unbounded above. We assume $\mathbb{T}$ has the topology that it inherites from the standard topology on $\mathbb{R}$. We assume throughout that p and q are continuous real valued functions on $\mathbb{T}$ with $p(t) > 0$ on $\mathbb{T}$ (in some papers it is only assumed that $p(t) \neq 0$ on $\mathbb{T}$).

Definition 7.1. For each $t \in \mathbb{T}$ the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

If $\sigma(t) > t$, we say t is right–scattered, while if $\rho(t) < t$ we say t is left–scattered. If $\sigma(t) = t$ we say t is right–dense, while if $\rho(t) = t$ we say t is left–dense.

If $x : \mathbb{T} \to \mathbb{R}$, then we define the function $x^{\sigma} : \mathbb{T} \to \mathbb{R}$ as

$$x^{\sigma}(t) := x(\sigma(t)), \qquad t \in \mathbb{T}.$$

Furthermore, we define an interval in $\mathbb{T}$ by

$$[a, \infty) := \{t \in \mathbb{T} : \ a \leq t\}.$$

Other types of intervals are defined similarly.

We are concerned with calculus on time scales which is a unified approach to continuous and discrete calculus. An excellent introduction is given by S. Hilger [33]. Agarwal and Bohner [1] refer to it as calculus on time scales. Other papers in this area include Agarwal and Bohner [2], Agarwal, Bohner, and Wong [3], Hilger and Erbe [24], and Erbe and Peterson [25, 26].

Definition 7.2. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$, then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\right| < \varepsilon \left|\sigma(t) - s\right|$$

for all $s \in U$. We call $x^{\Delta}(t)$ the delta derivative of $x(t)$.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right–scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, then

$$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t).$$

In particular if $\mathbb{T} = \mathbb{Z}$, then the equation (7.1) is the second order self–adjoint difference equation

$$Lx(t) = \Delta[p(t)\Delta x(t)] + q(t)x(t+1) = 0,$$

which as we have pointed out earlier is equivalent to a discrete Hamiltonian system.

Of course if $\mathbb{T} = \mathbb{R}$, then the expression (7.1) reduces to the second order differential equation

$$Lx(t) = [p(t)x'(t)]' + q(t)x(t) = 0,$$

which is equivalent to a continuous Hamiltonian system.

Definition 7.3. We say that $x : \mathbb{T} \to \mathbb{R}$ is right–dense continuous provided x is right continuous at all right–dense points in $\mathbb{T}$ and at all left–dense points $t \in \mathbb{T}$ the left hand limit $\lim_{s \to t^-} x(s)$ exists and is finite.

Definition 7.4. We say that $x : \mathbb{T} \to \mathbb{R}$ is in $\mathbb{D}$ provided x and px^{Δ} are continuous on $\mathbb{T}$ and $[px^{\Delta}]^{\Delta}$ is right–dense continuous on $\mathbb{T}$. We say that $x(t)$ is a solution of $Lx(t) = 0$ on $\mathbb{T}$ provided $x \in \mathbb{D}$ and $Lx(t) = 0$ for all $t \in \mathbb{T}$.

We now give a simple example given by Erbe and Peterson [25].

Example 7.5. Consider the differential equation

$$x^{\Delta\Delta}(t) + \pi^2 x(\sigma(t)) = 0,$$

on the measure chain

$$\mathbb{T} = [0,1] \cup \bigcup_{n=2}^{\infty} \{n\}.$$

Let x be the solution of the above equation satisfying the initial conditions $x(0) = 0$, $x^{\Delta}(0) = 1$. One can show that

$$x(t) = \begin{cases} \frac{1}{\pi}\sin \pi t & \text{if } 0 \le t \le 1, \\ -1 & \text{if } t = 2, \\ (2 - \pi^2)x(t-1) - x(t-2) & \text{if } t \ge 3. \end{cases}$$

Note that our solution "pieces" together a solution of a differential equation and a solution of a difference equation.

Definition 7.6. If $F^{\Delta}(t) = f(t)$, then we define $\int_a^t f(\tau)\,\Delta\tau = F(t) - F(a)$.

We will use elementary properties of this integral that are either available in [1, 33, 39] or are easy to verify.

Definition 7.7. We say $x(t,s)$ is the Cauchy function for $Lx = 0$ provided for each fixed $s \in \mathbb{T}$, $x(t,s)$ is the solution of the initial value problem

$$Lx(t,s) = 0, \quad x(\sigma(s),s) = 0, \quad x^{\Delta}(\sigma(s),s) = \frac{1}{p(\sigma(s))}.$$

Example 7.8. The Cauchy function for $Lx = [p(t)x^{\Delta}(t)]^{\Delta} = 0$ is given by

$$x(t,s) = \int_{\sigma(s)}^{t} \frac{1}{r(\tau)}\Delta\tau.$$

Theorem 7.9 (Variation of Constants)**.** *Suppose h is continuous on $[a,b]$ and $x(t,s)$ is the Cauchy function for $Lx(t) = 0$, then it follows that*

$$x(t) := \int_{a}^{t} x(t,s)h(s)\Delta s$$

is the solution of the initial value problem

$$Lx(t) = h(t), \quad x(a) = 0, \quad x^{\Delta}(a) = 0.$$

Definition 7.10. We define the Wronskian of x and y by

$$w[x(t),y(t)] = \begin{vmatrix} x(t) & y(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{vmatrix}.$$

Theorem 7.11. *For $t \in \mathbb{T}$,*

$$w[x(t),y(t)] = \begin{vmatrix} x^{\sigma}(t) & y^{\sigma}(t) \\ x^{\Delta}(t) & y^{\Delta}(t) \end{vmatrix}.$$

Theorem 7.12 (Lagrange Identity)**.** *If $y, z \in \mathbb{D}$, then*

$$z(\sigma(t))Ly(t) - y(\sigma(t))Lz(t) = \{z(t); y(t)\}^{\Delta},$$

where $\{z(t); y(t)\}$, called the Lagrange bracket of z and y, is defined by

$$\{z(t); y(t)\} = p(t)w[z(t), y(t)].$$

Definition 7.13. We define the inner product of x and y on $[\sigma(a), \sigma(b)]$ by

$$\langle x, y\rangle = \int_{\sigma(a)}^{\sigma(b)} x(t)y(t)\Delta t,$$

and we say x and y are orthogonal on $[\sigma(a), \sigma(b)]$ provided $\langle x, y\rangle = 0$.

The next result follows from integrating both sides of the Lagrange identity from a to b and using the definition of the inner product.

Theorem 7.14 (Green's Theorem)**.** *If $x, y \in \mathbb{D}$, then*

$$\langle z^{\sigma}, Ly\rangle - \langle y^{\sigma}, Lz\rangle = \{z(t); y(t)\}_a^b.$$

The following result follows immediately from the Lagrange identity.

Theorem 7.15 (Abel's Formula)**.** *If x, y are solutions of $Lx = 0$, then*

$$w[x(t), y(t)] = \frac{C}{p(t)}$$

for $t \in [a, \sigma(b)]$, where C is a constant.

The following result follows immediately from Abel's formula.

Corollary 7.16. *If x, y are solutions of $Lx = 0$, then either*

$$w[x(t), y(t)] = 0 \quad \text{for all} \quad t \in [a, \sigma(b)]$$

or

$$w[x(t), y(t)] \neq 0 \quad \text{for all} \quad t \in [a, \sigma(b)].$$

We now state a theorem that gives a formula for the Cauchy function for (7.1).

Theorem 7.17. *If u and v are linearly independent solutions of (7.1), then the Cauchy function $x(t, s)$ for (7.1) is given by*

$$x(t, s) = \frac{u(\sigma(s))v(t) - v(\sigma(s))u(t)}{p(\sigma(s))[u(s)v^\Delta(s) - u^\Delta(s)v(s)]}.$$

Definition 7.18. Let $a, b \in \mathbb{T}$. We want to consider $Lx(t) = 0$ on the interval $[a, \sigma^2(b)]$. We say a nontrivial solution of $Lx = 0$ has a generalized zero at a iff $x(a) = 0$. We say a nontrivial solution x has a generalized zero at $t_0 \in (a, \sigma^2(b)]$ provided either $x(t_0) = 0$ or $x(\rho(t_0))x(t_0) < 0$. Finally we say $Lx = 0$ is disconjugate on $[a, \sigma^2(b)]$ provided there is no nontrivial solution of $Lx = 0$ with two (or more) generalized zeros in $[a, \sigma^2(b)]$.

Theorem 7.19. *If $Lx = 0$ is disconjugate on $[a, \sigma^2(b)]$, then $Lx = 0$ has a positive solution on $[a, \sigma^2(b)]$.*

We finish this paper by stating a Reid roundabout theorem (see Theorem 3.4) for the time scales case. Work on the full Reid roundabout theorem for the case of linear Hamiltonian systems on time scales

$$x^\Delta = A(t)x^\sigma + B(t)u, \qquad u^\Delta = C(t)x^\sigma - A^*(t)u$$

is currently in progress (see [35, 36]); however, in [2, Theorem 5] a Reid roundabout theorem for the case of self–adjoint vector difference equations

$$\left[R(t)Y^\Delta\right]^\Delta + P(t)Y^\sigma = 0$$

is established. It reads as follows.

Theorem 7.20 (Jacobi's Condition on Time Scales). *Let R and P be right–dense continuous $n \times n$-matrix valued functions on $\mathbb{T}$ such that $R(t)$ and $P(t)$ are symmetric and have real entries and such that $R(t)$ is invertible for each $t \in \mathbb{T}$. Let $a, b \in \mathbb{T}$ with $a < \rho(b)$. Then the quadratic functional*

$$\mathcal{F}_T(y) = \int_a^b \left\{ (y^\Delta)^T R y^\Delta - (y^\sigma)^T P y^\sigma \right\} (t) \Delta t$$

is positive definite, i.e., $\mathcal{F}_T(y) > 0$ for all nontrivial piecewise differentiable functions with right–dense continuous derivative $y : [a,b] \cap \mathbb{T} \to \mathbb{R}^n$ with $y(a) = y(b) = 0$, if and only if the solution Y of

$$\left[R(t)Y^\Delta\right]^\Delta + P(t)Y^\sigma = 0, \qquad Y(a) = 0, \qquad Y^\Delta(a) = R^{-1}(a)$$

satisfies

$$\begin{cases} Y(t) \quad \text{invertible for all} \quad t \in (a,b] \cap \mathbb{T}, \quad \text{and} \\ Y(t)Y^{-1}(\sigma(t))R^{-1}(t) \quad \text{positive definite for all} \quad t \in (a, \rho(b)] \cap \mathbb{T}. \end{cases}$$

The reader may compare how the continuous and the discrete strengthened Jacobi conditions appear when substituting $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively, in the above theorem. In particular it is interesting to realize how the strengthened Legendre condition $R(t) > 0$ appears when translating the above theorem for $\mathbb{T} = \mathbb{R}$.

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