Oscillation of a Family of q-Difference Equations

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Abstract. We obtain the complete classification of oscillation and nonoscillation for the q-difference equation

\[ x^{\Delta\Delta}(t) + \frac{b(-1)^n}{c} x(qt) = 0, \quad b \neq 0, \]

where \( t = q^n \in \mathbb{T} = q^\mathbb{Z}, q > 1, c, b \in \mathbb{R} \). In particular we prove that this q-difference equation is nonoscillatory, if \( c > 2 \) and is oscillatory, if \( c < 2 \). In the critical case \( c = 2 \) we show that it is oscillatory, if \( |b| > \frac{1}{q(1-q)} \), and is nonoscillatory, if \( |b| \leq \frac{1}{q(1-q)} \).

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1. Introduction

Let \( \mathbb{T} \) be a time scale (i.e., a closed nonempty subset of \( \mathbb{R} \)) with \( \sup \mathbb{T} = \infty \). Consider the second order dynamic equation on time scale

\[ x^{\Delta\Delta}(t) + p(t)x^{\sigma}(t) = 0, \]

where \( \sigma \) is the jump operator and \( f^{\sigma} = f \circ \sigma \) (composition of \( f \) with \( \sigma \)), \( p \) is right-dense continuous functions on \( \mathbb{T} \) and

\[ \int_{t_0}^{\infty} p(t)\Delta t := \lim_{t \to \infty} \int_{t_0}^{t} p(s)\Delta s \quad \text{exists (finite)}. \]

When \( \mathbb{T} = \mathbb{R} \) the dynamic equation (1.1) is the differential equation

\[ x'' + p(t)x = 0, \]
and when $T = Z$ the dynamic equation (1.1) is the difference equation

$$
\Delta^2 x(t) + p(t)x^\sigma(t) = 0.
$$

When $T = q^{^N}$, $q > 1$, the dynamic equations (1.1) are called $q$-difference equations, which have important applications in quantum theory [8]. Our main results are for a family of $q$-difference equations. For $T = R$, in [10] and [4], Willett and Wong proved, respectively, the following theorems.

**Theorem A.** (Willett-Wong, [10], [4]) Suppose that

$$
\int_t^\infty \bar{P}(s)P(s,t)ds \leq \frac{1}{4}\bar{P}(t),
$$

for large $t$, where $\bar{P}(t) = \int_t^\infty P^2(s)Q_P(s,t)ds$, $Q_P(s,t) = \exp\left(2\int_s^t P(\tau)d\tau\right)$. Then the differential equation (1.2) is nonoscillatory.

**Theorem B.** (Willett-Wong, [10], [4]) If $\bar{P}(t) \neq 0$ satisfies

$$
\int_t^\infty P^2(s)Q_P(s,t)ds \geq \frac{1+\epsilon}{4}\bar{P}(t),
$$

for some $\epsilon > 0$ and large $t$. Then the differential equation (1.2) is oscillatory.

As applications of Theorems A and B, Willett [10] considered the very sensitive differential equation

$$
x'' + \mu \sin \nu t \eta x = 0
$$

for $|\frac{\mu}{\nu}| \neq \frac{1}{\sqrt{2}}$, $\mu \neq 0, \nu \neq 0, \eta$ constants and proved that (1.4) is nonoscillatory, if $\eta > 1$ and is oscillatory, if $\eta < 1$. When $\eta = 1$, (1.4) is oscillatory, if $|\frac{\mu}{\nu}| > \frac{1}{\sqrt{2}}$, and is nonoscillatory, if $|\frac{\mu}{\nu}| < \frac{1}{\sqrt{2}}$.

Wong proved the following very nice result.

**Theorem C.** (Wong, [4]) If there exists a functions $\bar{B}(t)$ such that

$$
\int_t^\infty [\bar{P}(s) + \bar{B}(s)]^2Q_P(s,t)ds \leq \bar{B}(t),
$$

for large $t$, then the differential equation (1.2) is nonoscillatory.

As applications of Theorem C, Wong proved that the equation (1.4) is nonoscillatory, for $|\frac{\mu}{\nu}| = \frac{1}{\sqrt{2}}$.

In [1], [2], we extended Theorems A, B, and C to the time scale case using a so-called ‘second-level Riccati equation’ (see [3] for the discrete case) or what Wong refers to as a new Riccati integral equation in the continuous case. Using this approach, one is able to handle various critical cases. These ideas are novel in treating the case when $P(t) := \int_t^\infty p(s)ds$ is not of one sign for large $t$.

A special case of results in [1] and [2], is that the difference equation

$$
\Delta^2 x(n) + \frac{b(-1)^n}{n^c}x(n + 1) = 0, \quad b \neq 0,
$$

where $b, c \in \mathbb{R}$ is nonoscillatory, if $c > 1$ and is oscillatory, if $c < 1$. Also if $c = 1$, then (1.5) is oscillatory, if $|b| > 1$ and is nonoscillatory, if $|b| \leq 1$. 


Lemma 1.1. [2, Theorem 3.2] Assume that \( \int_0^\infty p(t) \Delta t \) is convergent, 
\( P(t) = \int_1^\infty p(s) \Delta s, \) \( 1 \pm \mu(t) P(t) > 0, \) for large \( t. \) If \( \int_T^\infty P^2(t) \times e^{-p(t,T)} \Delta t \) is convergent and
\[
(1.6) \quad P(t) := \int_t^\infty e^{2P_{(s,t)}} \frac{P^2(s)}{1 - \mu(s)P(s)} \Delta s
\]
satisfies
\[
(1.7) \quad \frac{1}{4} P(t) \geq \int_t^\infty e^{2P_{(s,t)}} \frac{P(s)P(\sigma(s))}{1 - \mu(s)P(s)} \Delta s.
\]
for large \( t, \) then (1.1) is nonoscillatory.

2. Main Theorem

Our main concern in this paper is the \( q \)-difference equation
\[
(2.1) \quad x^{\Delta \Delta}(t) + \frac{b(-1)^n}{t^c} x(qt) = 0, \quad b \neq 0,
\]
where \( t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}, q > 1, b, c \in \mathbb{R} \) and our main result is the following complete classification of (2.1). Since the graininess function for \( \mathbb{T} = q^{\mathbb{N}_0} \) is unbounded, we can not use Theorem 4.1 in [2], when we consider the oscillation of the \( q \)-difference equation (2.1).

**Theorem 2.1.** The \( q \)-difference equation (2.1) is nonoscillatory, if \( c > 2, \) and is oscillatory, if \( c < 2. \) If \( c = 2, \) then (2.1) is oscillatory, if \( |b| > \frac{1}{q(q-1)}, \) and is nonoscillatory, if \( |b| \leq \frac{1}{q(q-1)}. \)

**Proof.** First consider the case \( c > 2. \) Note that for \( t = q^{2k} \)
\[
P(t) = \int_t^\infty p(\tau) \Delta \tau = \sum_{j=2k}^{\infty} p(q^j) \mu(q^j)
\]
\[
= \frac{b(q-1)q^{2k}}{q^{2kc}} \left[ 1 - \frac{q}{q^c} + \frac{q^2}{q^{2c}} - \ldots \right]
\]
\[
= b \frac{q^{c-1}(q-1)}{q^{2kc}(c-1)(q^{c-1}+1)}.
\]
Similarly, we have
\[
P(q^{2k+1}) = -b \frac{q^{c-1}(q-1)}{q^{(2k+1)c-1}(q^{c-1}+1)}
\]
and hence in general
\[
(2.2) \quad P(t) = P(t^n) = b \frac{(-1)^n q^{c-1}(q-1)}{q^{nc-1}(q^{c-1}+1)} = b \frac{(-1)^n q^{c-1}(q-1)}{t^{c-1}(q^{c-1}+1)}.
\]
Since \( c > 2, \) we get that
\[
\lim_{t \to \infty} \mu(t) P(t) = \lim_{n \to \infty} b \frac{(-1)^n q^{c-1}(q-1)^2}{t^{c-2}(q^{c-1}+1)} = 0,
\]
which implies that for large $t$, $\pm P$ are positively regressive.

By the definition of the exponential [5, Definition 2.30] we have for $s \geq t$

$$e_{\pm P}(s, t) = \exp \int_t^s \frac{1}{\tau(q-1)} \ln \left( 1 \pm \frac{b(q-1)^2(-1)^{\ln \tau}}{\tau^{q-2}(1 + q^{1-c})} \right) \Delta \tau$$

(2.3)

$$= \exp \left[ \sum_{i=n}^{m-1} \ln \left( 1 \pm \frac{b(q-1)^2(-1)^i}{q^i(q^{c-2})(1 + q^{1-c})} \right) \right].$$

Note that $\ln(1 \pm x) \sim \pm x$, so when $c > 2$, the two series

(2.4)

$$\sum_{i=n}^{\infty} \ln \left( 1 \pm \frac{b(q-1)^2(-1)^i}{q^i(q^{c-2})(1 + q^{1-c})} \right).$$

are absolutely convergent.

Using properties of the exponential [5, Theorem 2.36], we have

$$e_{\frac{1}{2} \mu P}(s, t) = \frac{e_P(s, t)}{e_{-P}(s, t)}.$$

By (2.3), (2.4) and $\lim_{t \to \infty} \mu(t)P(t) = 0$, given $0 < \epsilon < 1$, there exists a large $N$, so that when $s = q^m \geq t = q^n \geq q^N$,

(2.5)

$$1 - \epsilon \leq e_{\frac{1}{2} \mu P}(s, t) \frac{1}{1 - \mu(s)P(s)} \leq 1 + \epsilon.$$

So from (2.2), we get that

$$\tilde{P}(t) = \int_t^\infty e_{\frac{1}{2} \mu P}(s, t) \frac{P^2(s)}{1 - \mu(s)P(s)} \Delta s \leq (1 + \epsilon) \int_t^\infty P^2(s) \Delta s$$

$$\leq (1 + \epsilon) b^2 \frac{[q^{c-1}(q-1)]^2}{(q^{c-1} + 1)^2} \sum_{i=n}^{\infty} q^i(q-1) \frac{1}{q^{2i(c-1)}}$$

(2.6)

$$= (1 + \epsilon) b^2 \frac{q^{2(c-1)}(q-1)^3}{(q^{c-1} + 1)^2} \cdot \frac{q^{2(c-1)}}{q^{2c(c-1)} - q} \left[ \frac{q}{q^{2(c-1)}} \right]^n,$$

for large $t$. It follows that

$$\tilde{P}(\sigma(t)) \leq (1 + \epsilon) b^2 \frac{q^{2(c-1)}(q-1)^3}{(q^{c-1} + 1)^2} \cdot \frac{q^{2(c-1)}}{q^{2(c-1)} - q} \left[ \frac{q}{q^{2(c-1)}} \right]^{n+1}.$$
which implies that (2.9) is oscillatory. Similarly, when

\[ b > \frac{q+1}{q(q-1)^2}, \]

(2.10) is oscillatory.

So

\[
\int_t^\infty e^{\frac{2\mu}{1-\mu}} (s,t) \frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)} \Delta s \\
\leq (1+\epsilon)^3 b^4 \left[ \frac{q^2(c-1)(q-1)^3}{(q^c-1)^2} \cdot \frac{q^2(c-1)-q}{q^2(c-1)} \right]^{2} \\
\times \sum_{i=n}^{\infty} \left[ \frac{q^{i+1}}{q^{2(i+1)(c-1)}} \cdot \frac{q^i}{q^{2i(c-1)}} q'(q-1) \right] \\
(2.7) = (1+\epsilon)^3 b^4 \left[ \frac{q^4(c-1)(q-1)^2}{(q^c-1)^4} \cdot \frac{q^2(c-1)-q}{q^2(c-1)} \right]^{2} \frac{q^{3n+1}}{q^{4n+2(c-1)}}. \\
\]

Similar to the proof of (2.6), we also have

\[
(2.8) \quad \frac{1}{4} \bar{P}(t) > \frac{(1-\epsilon)b^2}{4} \cdot \frac{q^2(c-1)(q-1)^3}{(q^c-1)^2} \cdot \frac{q^2(c-1)-q}{q^2(c-1)} \left[ \frac{q}{q^2(c-1)} \right]^n, \\
\]

for large \( t \). Note that when \( c > 2 \),

\[
\lim_{n\to\infty} \frac{q^{3n+1}}{q^{4n+2(c-1)}} = 0. \\
\]

From (2.7), (2.8), we have that, for sufficiently large \( t \),

\[
\int_t^\infty e^{\frac{2\mu}{1-\mu}} (s,t) \frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)} \Delta s < \frac{1}{4} \bar{P}(t). \\
\]

By Lemma 1.1, equation (2.1) is nonoscillatory.

Next we consider the case \( c = 2 \), that is we consider

(2.9)

\[ x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^2} x(qt) = 0 \]

where \( t = q^n \in T = q^{N_0}, q > 1 \). Expanding out equation (2.9) we obtain

(2.10)

\[ x(q^{n+2}) - [q + 1 - bq(q-1)^2(-1)^n] x(q^{n+1}) + qx(q^n) = 0. \]

When \( b \neq \frac{q+1}{q(q-1)^2} \), we get from (2.10) when \( n = 2k \) is even \( x(q^{2k+2}) = -qx(q^{2k}) \), which implies that (2.10) is oscillatory. Similarly, when \( b = -\frac{q+1}{q(q-1)^2}, (2.10) \) is also oscillatory.

Let \( d_n = q + 1 - bq(q-1)^2(-1)^n \) in equation (2.10). If we suppose that

\[ b > -\frac{q+1}{q(q-1)^2}, \]

(2.11) \[ x(q^{2k+2}) + qx(q^{2k}) = d_k x(q^{2k+1}). \]

which implies that (2.9) is oscillatory. Similarly, when \( b < -\frac{q+1}{q(q-1)^2}, (2.10) \) is also oscillatory.
Therefore in the following, we can assume that $|b| < \frac{q+1}{q(q-1)}$, so we have $d_n > 0$. Assume $x(t) = x(q^n)$ is a solution of (2.10) satisfying $x(t) = x(q^n) \neq 0$ for all large $n$. Then from (2.10), we get that
\[ \frac{q}{d_{n+1}d_n} \cdot \frac{d_{n+1}x(q^{n+2})}{q x(q^{n+1})} + \frac{q x(q^n)}{d_n x(q^{n+1})} = 1. \]
Let $y(n) := \frac{d_n x(q^{n+1})}{q x(q^n)}$ and $A := \frac{q}{d_{n+1}d_n} = \frac{q}{(q+1)^2 - b^2(q(q-1)^2} > 0$ is a positive constant. We get
\[ Ay(n + 1) + \frac{1}{y(n)} = 1. \]
Letting $y(n) = \frac{z(n+1)}{z(n)}$, we get the second order difference equation
\[ Az(n + 2) - z(n + 1) + z(n) = 0. \]
The characteristic equation of (2.13) is $\lambda^2 - \frac{1}{A} \lambda + \frac{1}{A} = 0$.

When $\frac{1-4A}{A} < 0$, that is $|b| > \frac{1}{q(q-1)}$, the characteristic equation of (2.13) has complex roots $\lambda = re^{i\theta}, \theta \neq k\pi, k$ an integer. So (2.13) has an oscillatory solution $z(n) = r^n \sin n\theta$. This means $y(n) = \frac{z(n+1)}{z(n)} = \frac{r \sin(n+1)\theta}{\sin n\theta}$ is an oscillatory solution of (2.12). Noticing that $d_n > 0$ and $y(n) = \frac{d_n x(q^{n+1})}{q x(q^n)}$, we get that (2.10) has an oscillatory solution. Hence, we get that (2.10) is oscillatory.

When $\frac{1-4A}{A} \geq 0$, that is $|b| \leq \frac{1}{q(q-1)}$, the characteristic equation of (2.13) has a real root $\lambda = \frac{1+\sqrt{1-4A}}{2A} > 0$. So (2.13) has a nonoscillatory solution $z(n) = \lambda^n > 0$. This means $y(n) = \frac{z(n+1)}{z(n)} = \lambda > 0$ is a nonoscillatory solution of (2.12). Noticing that $d_n > 0$ and $y(n) = \frac{d_n x(q^n)}{q x(q^n)}$, we get that (2.10) has a nonoscillatory solution. Hence, we get that (2.10) is nonoscillatory.

**Remark** As in the case $c > 2$, using Lemma 1.1, we can also prove that (2.10) is nonoscillatory, when $|b| \leq \frac{1}{q(q-1)}$, but we can not use Theorem 4.1 in [2] to prove the oscillation of (2.10) when $|b| > \frac{1}{q(q-1)}$, since the graininess function of $q^{N_0}$ is unbounded.

Finally we consider the $q$-difference equation for the case $c < 2$.
\[ x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^c} x(qt) = 0 \]
where $t = q^n \in T = q^{N_0}, q > 1, b \neq 0, c < 2$.

To show that (2.14) is oscillatory, for all $c < 2$, we need the following useful comparison theorem [7].

**Theorem 2.2.** Assume $a \in C_{rd}^1, a(t) \geq 1, \mu(t)a^{\Delta}(t) \geq 0$ and $a^{\Delta\Delta}(t) \leq 0$. Then (1.1) is oscillatory implies $x^{\Delta\Delta}(t) + a(t)p(t)x(\sigma(t)) = 0$ is oscillatory on $[t_0, \infty)$.
Letting $b_0 := \frac{q+1}{q(q-1)^2} > \frac{1}{q(q-1)}$, we have by Theorem 2.1, that
\[ x^\Delta(t) \pm b_0 \frac{(-1)^n}{t^2} x(qt) = 0 \]
is oscillatory. Let $a(t) = At^\alpha$, $A > 0$, $0 < \alpha < 1$. We have $a(t) \geq 1$, for large $t$ and $a^\Delta(t) \geq 0$. It is easy to get that
\[ a^\Delta(t) = \frac{At^\alpha(q^\alpha - 1)(q^\alpha - q)}{t^2q(q-1)^2} \leq 0. \]
Repeated applications of Theorem 2.2, gives us that
\[ x^\Delta(t) \pm Bt^\beta b_0 \frac{(-1)^n}{t^2} x(qt) = 0 \]
is oscillatory, for all $\beta > 0$, $B > 0$. So the equation
\[ x^\Delta(t) \pm Bb_0 \frac{(-1)^n}{t^{2-\beta}} x(qt) = 0 \]
is oscillatory, for all $\beta > 0$, $B > 0$. This means that the equation
\[ x^\Delta(t) + b \frac{(-1)^n}{t^c} x(qt) = 0 \]
is oscillatory, for $b \neq 0$, $c < 2$. □

References