Oscillation of a Family of q-Difference Equations

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Abstract. We obtain the complete classification of oscillation and nonoscillation for the q-difference equation

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^c} x(qt) = 0, \quad b \neq 0,$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $c, b \in \mathbb{R}$. In particular we prove that this q-difference equation is nonoscillatory, if $c > 2$ and is oscillatory, if $c < 2$. In the critical case $c = 2$ we show that it is oscillatory, if $|b| > \frac{1}{q(q-1)}$, and is nonoscillatory, if $|b| \leq \frac{1}{q(q-1)}$.
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1. Introduction

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. Consider the second order dynamic equation on time scale

$$x^{\Delta\Delta}(t) + p(t)x^{\sigma}(t) = 0, \tag{1.1}$$

where σ is the jump operator and $f^{\sigma} = f \circ \sigma$ (composition of f with σ), p is right-dense continuous functions on $\mathbb{T}$ and

$$\int_{t_0}^{\infty} p(t)\Delta t := \lim_{t\to\infty} \int_{t_0}^{t} p(s)\Delta s \qquad \text{exists (finite).}$$

When $\mathbb{T} = \mathbb{R}$ the dynamic equation (1.1) is the differential equation

$$x'' + p(t)x = 0, \tag{1.2}$$

and when $\mathbb{T}=\mathbb{Z}$ the dynamic equation (1.1) is the difference equation

$$\Delta^2 x(t)+p(t)x^\sigma(t)=0. \tag{1.3}$$

When $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, the dynamic equations (1.1) are called q-difference equations, which have important applications in quantum theory [**8**]. Our main results are for a family of q-difference equations. For $\mathbb{T}=\mathbb{R}$, in [**10**] and [**4**], Willett and Wong proved, respectively, the following theorems.

Theorem A.(Willett-Wong, [**10**], [**4**]) Suppose that

$$\int_t^\infty \bar{P}^2(s)Q_P(s,t)ds \le \frac{1}{4}\bar{P}(t),$$

for large t, where $\bar{P}(t)=\int_t^\infty P^2(s)Q_P(s,t)ds$, $Q_P(s,t)=\exp\left(2\int_t^s P(\tau)d\tau\right)$. Then the differential equation (1.2) is nonoscillatory.

Theorem B.(Willett-Wong, [**10**], [**4**]) If $\bar{P}(t)\not\equiv 0$ satisfies

$$\int_t^\infty \bar{P}^2(s)Q_P(s,t)ds \ge \frac{1+\epsilon}{4}\bar{P}(t),$$

for some $\epsilon>0$ and large t. Then the differential equation (1.2) is oscillatory.

As applications of Theorems A and B, Willett [**10**] considered the very sensitive differential equation

$$x''+\frac{\mu\sin\nu t}{t^\eta}x=0 \tag{1.4}$$

for $|\frac{\mu}{\nu}|\neq\frac{1}{\sqrt{2}}$, $\mu\neq 0,\nu\neq 0,\eta$ constants and proved that (1.4) is nonoscillatory, if $\eta>1$ and is oscillatory, if $\eta<1$. When $\eta=1$, (1.4) is oscillatory, if $|\frac{\mu}{\nu}|>\frac{1}{\sqrt{2}}$, and is nonoscillatory, if $|\frac{\mu}{\nu}|<\frac{1}{\sqrt{2}}$.

Wong proved the following very nice result.

Theorem C.(Wong, [**4**]) If there exists a functions $\bar{B}(t)$ such that

$$\int_t^\infty [\bar{P}(s)+\bar{B}(s)]^2 Q_P(s,t)ds \le \bar{B}(t),$$

for large t, then the differential equation (1.2) is nonoscillatory.

As applications of Theorem C, Wong proved that the equation (1.4) is nonoscillatory, for $|\frac{\mu}{\nu}|=\frac{1}{\sqrt{2}}$.

In [**1**],[**2**], we extended Theorems A, B, and C to the time scale case using a so-called 'second-level Riccati equation' (see [**3**] for the discrete case) or what Wong refers to as a new Riccati integral equation in the continuous case. Using this approach, one is able to handle various critical cases. These ideas are novel in treating the case when $P(t):=\int_t^\infty p(s)ds$ is not of one sign for large t.

A special case of results in [**1**] and [**2**], is that the difference equation

$$\Delta^2 x(n)+\frac{b(-1)^n}{n^c}x(n+1)=0, \qquad b\neq 0, \tag{1.5}$$

where $b,c\in\mathbb{R}$ is nonoscillatory, if $c>1$ and is oscillatory, if $c<1$. Also if $c=1$, then (1.5) is oscillatory, if $|b|>1$ and is nonoscillatory, if $|b|\le 1$.

LEMMA 1.1. [**2**, Theorem 3.2] *Assume that* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent,* $P(t) = \int_t^{\infty} p(s)\Delta s$, $1 \pm \mu(t)P(t) > 0$, *for large* t. *If* $\int_T^{\infty} P^2(t) \times \frac{e_P(t,T)}{e_{-P}(t,T)}\Delta t$ *is convergent and*

$$\bar{P}(t) := \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s \tag{1.6}$$

satisfies

$$\frac{1}{4}\bar{P}(t) \geq \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s. \tag{1.7}$$

for large t, *then* (1.1) *is nonoscillatory.*

2. Main Theorem

Our main concern in this paper is the q-difference equation

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^c}x(qt) = 0, \quad b \neq 0, \tag{2.1}$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}, q > 1$, $b, c \in \mathbb{R}$ and our main result is the following complete classification of (2.1). Since the graininess function for $\mathbb{T} = q^{\mathbb{N}_0}$ is unbounded, we can not use Theorem 4.1 in [**2**], when we consider the oscillation of the q-difference equation (2.1).

THEOREM 2.1. *The q-difference equation* (2.1) *is nonoscillatory, if* $c > 2$, *and is oscillatory, if* $c < 2$. *If* $c = 2$, *then* (2.1) *is oscillatory, if* $|b| > \frac{1}{q(q-1)}$, *and is nonoscillatory, if* $|b| \leq \frac{1}{q(q-1)}$.

PROOF. First consider the case $c > 2$. Note that for $t = q^{2k}$

$$\begin{aligned} P(t) &= \int_t^{\infty} p(\tau)\Delta\tau = \sum_{j=2k}^{\infty} p(q^j)\mu(q^j) \\ &= \frac{b(q-1)q^{2k}}{q^{2kc}}\left[1 - \frac{q}{q^c} + \frac{q^2}{q^{2c}} - \cdots\right] \\ &= b\,\frac{q^{c-1}(q-1)}{q^{2k(c-1)}(q^{c-1}+1)}. \end{aligned}$$

Similarly, we have

$$P(q^{2k+1}) = -b\,\frac{q^{c-1}(q-1)}{q^{(2k+1)(c-1)}(q^{c-1}+1)}$$

and hence in general

$$P(t) = P(t^n) = b\,\frac{(-1)^n q^{c-1}(q-1)}{q^{n(c-1)}(q^{c-1}+1)} = b\,\frac{(-1)^n q^{c-1}(q-1)}{t^{c-1}(q^{c-1}+1)}. \tag{2.2}$$

Since $c > 2$, we get that

$$\lim_{t\to\infty} \mu(t)P(t) = \lim_{n\to\infty} b\frac{(-1)^n q^{c-1}(q-1)^2}{t^{c-2}(q^{c-1}+1)} = 0,$$

which implies that for large t, $\pm P$ are positively regressive.

By the definition of the exponential [**5**, Definition 2.30] we have for $s \geq t$

$$\begin{aligned} e_{\pm P}(s,t) &= \exp \int_t^s \frac{1}{\tau(q-1)} \ln\left(1 \pm \frac{b(q-1)^2(-1)^{\frac{\ln\tau}{\ln q}}}{\tau^{c-2}(1+q^{(1-c)})}\right)\Delta\tau \\ &= \exp\left[\sum_{i=n}^{m-1} \ln\left(1 \pm \frac{b(q-1)^2(-1)^i}{q^{i(c-2)}(1+q^{1-c})}\right)\right]. \qquad (2.3) \end{aligned}$$

Note that $\ln(1 \pm x) \sim \pm x$, so when $c > 2$, the two series

$$\sum_{i=n}^{\infty} \ln\left(1 \pm \frac{b(q-1)^2(-1)^i}{q^{i(c-2)}(1+q^{(1-c)})}\right). \qquad (2.4)$$

are absolutely convergent.

Using properties of the exponential [**5**, Theorem 2.36], we have

$$e_{\frac{2P}{1-\mu P}}(s,t) = \frac{e_P(s,t)}{e_{-P}(s,t)}.$$

By (2.3), (2.4) and $\lim_{t\to\infty} \mu(t)P(t) = 0$, given $0 < \epsilon < 1$, there exists a large N, so that when $s = q^m \geq t = q^n \geq q^N$,

$$1 - \epsilon \leq e_{\frac{2P}{1-\mu P}}(s,t)\frac{1}{1-\mu(s)P(s)} \leq 1 + \epsilon. \qquad (2.5)$$

So from (2.2), we get that

$$\begin{aligned} \bar{P}(t) &= \int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s \leq (1+\epsilon)\int_t^\infty P^2(s)\Delta s \\ &\leq (1+\epsilon)b^2\frac{[q^{c-1}(q-1)]^2}{(q^{c-1}+1)^2}\sum_{i=n}^{\infty} q^i(q-1)\frac{1}{q^{2i(c-1)}} \\ &= (1+\epsilon)b^2\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\left[\frac{q}{q^{2(c-1)}}\right]^n, \qquad (2.6) \end{aligned}$$

for large t. It follows that

$$\bar{P}(\sigma(t)) \leq (1+\epsilon)b^2\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\left[\frac{q}{q^{2(c-1)}}\right]^{n+1}.$$

So

$$\begin{aligned}
&\int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s\\
\leq\ & (1+\epsilon)^3 b^4\left[\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\right]^2\\
\times\ & \sum_{i=n}^{\infty}\left[\frac{q^{i+1}}{q^{2(i+1)(c-1)}}\cdot\frac{q^i}{q^{2i(c-1)}}q^i(q-1)\right]\\
=\ & (1+\epsilon)^3 b^4\left[\frac{q^{4(c-1)}(q-1)^7}{(q^{c-1}+1)^4}\right]\cdot\left[\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\right]^2\frac{\frac{q^{3n+1}}{q^{(4n+2)(c-1)}}}{1-\frac{q^3}{q^{4(c-1)}}}. \qquad (2.7)
\end{aligned}$$

Similar to the proof of (2.6), we also have

$$\frac{1}{4}\bar{P}(t) > \frac{(1-\epsilon)b^2}{4}\cdot\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\left[\frac{q}{q^{2(c-1)}}\right]^n, \qquad (2.8)$$

for large t. Note that when $c > 2$,

$$\lim_{n\to\infty}\frac{\frac{q^{3n+1}}{q^{(4n+2)(c-1)}}}{\frac{q^n}{q^{2n(c-1)}}} = 0.$$

From (2.7), (2.8), we have that, for sufficiently large t,

$$\int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s < \frac{1}{4}\bar{P}(t).$$

By Lemma 1.1, equation (2.1) is nonoscillatory.

Next we consider the case $c = 2$, that is we consider

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^2}x(qt) = 0 \qquad (2.9)$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}, q > 1$. Expanding out equation (2.9) we obtain

$$x(q^{n+2}) - [q+1-bq(q-1)^2(-1)^n]x(q^{n+1}) + qx(q^n) = 0. \qquad (2.10)$$

When $b = \frac{q+1}{q(q-1)^2}$, we get from (2.10) when $n = 2k$ is even $x(q^{2k+2}) = -qx(q^{2k})$, which implies that (2.10) is oscillatory. Similarly, when $b = -\frac{q+1}{q(q-1)^2}$, (2.10) is also oscillatory.

Let $d_n = q+1-bq(q-1)^2(-1)^n$ in equation (2.10). If we suppose that $b > \frac{q+1}{q(q-1)^2}$, we have $d_{2k} < 0$. From (2.10), we get for $n = 2k$

$$x(q^{2k+2}) + qx(q^{2k}) = d_{2k}x(q^{2k+1}). \qquad (2.11)$$

which implies that (2.9) is oscillatory. Similarly, when $b < -\frac{q+1}{q(q-1)^2}$, (2.10) is also oscillatory.

Therefore in the following, we can assume that $|b| < \frac{q+1}{q(q-1)^2}$, so we have $d_n > 0$. Assume $x(t) = x(q^n)$ is a solution of (2.10) satisfying $x(t) = x(q^n) \neq 0$ for all large n. Then from (2.10), we get that

$$\frac{q}{d_{n+1}d_n} \cdot \frac{d_{n+1}x(q^{n+2})}{qx(q^{n+1})} + \frac{qx(q^n)}{d_n x(q^{n+1})} = 1.$$

Let $y(n) := \frac{d_n x(q^{n+1})}{qx(q^n)}$ and $A := \frac{q}{d_{n+1}d_n} = \frac{q}{(q+1)^2 - b^2q^2(q-1)^4} > 0$ is a positive constant. We get

$$Ay(n+1) + \frac{1}{y(n)} = 1. \tag{2.12}$$

Letting $y(n) = \frac{z(n+1)}{z(n)}$, we get the second order difference equation

$$Az(n+2) - z(n+1) + z(n) = 0. \tag{2.13}$$

The characteristic equation of (2.13) is $\lambda^2 - \frac{1}{A}\lambda + \frac{1}{A} = 0$.

When $\frac{1-4A}{A^2} < 0$, that is $|b| > \frac{1}{q(q-1)}$, the characteristic equation of (2.13) has complex roots $\lambda = re^{i\theta}$, $\theta \neq k\pi$, k an integer. So (2.13) has an oscillatory solution $z(n) = r^n \sin n\theta$. This means $y(n) = \frac{z(n+1)}{z(n)} = \frac{r\sin(n+1)\theta}{\sin n\theta}$ is an oscillatory solution of (2.12). Noticing that $d_n > 0$ and $y(n) = \frac{d_n x(q^{n+1})}{qx(q^n)}$, we get that (2.10) has an oscillatory solution. Hence, we get that (2.10) is oscillatory.

When $\frac{1-4A}{A^2} \geq 0$, that is $|b| \leq \frac{1}{q(q-1)}$, the characteristic equation of (2.13) has a real root $\lambda = \frac{1+\sqrt{1-4A}}{2A} > 0$. So (2.13) has a nonoscillatory solution $z(n) = \lambda^n > 0$. This means $y(n) = \frac{z(n+1)}{z(n)} = \lambda > 0$ is a nonoscillatory solution of (2.12). Noticing that $d_n > 0$ and $y(n) = \frac{d_n x(q^n)}{qx(q^{n+1})}$, we get that (2.10) has a nonoscillatory solution. Hence, we get that (2.10) is nonoscillatory.

Remark As in the case $c > 2$, using Lemma 1.1, we can also prove that (2.10) is nonoscillatory, when $|b| \leq \frac{1}{q(q-1)}$, but we can not use Theorem Theorem 4.1 in [**2**] to prove the oscillation of (2.10) when $|b| > \frac{1}{q(q-1)}$, since the graininess function of $q^{\mathbb{N}_0}$ is unbounded.

Finally we consider the q-difference equation for the case $c < 2$.

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^c}x(qt) = 0 \tag{2.14}$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}, q > 1$, $b \neq 0$, $c < 2$.

To show that (2.14) is oscillatory, for all $c < 2$, we need the following useful comparison theorem [**7**].

THEOREM 2.2. *Assume $a \in C^1_{rd}$, $a(t) \geq 1$, $\mu(t)a^{\Delta}(t) \geq 0$ and $a^{\Delta\Delta}(t) \leq 0$. Then* (1.1) *is oscillatory implies $x^{\Delta\Delta}(t) + a(t)p(t)x(\sigma(t)) = 0$ is oscillatory on* $[t_0, \infty)$.

Letting $b_0 := \frac{q+1}{q(q-1)^2} > \frac{1}{q(q-1)}$, we have by Theorem 2.1, that

$$x^{\Delta\Delta}(t) \pm b_0 \frac{(-1)^n}{t^2} x(qt) = 0$$

is oscillatory. Let $a(t) = At^\alpha$, $A > 0, 0 < \alpha < 1$. We have $a(t) \geq 1$, for large t and $a^\Delta(t) \geq 0$. It is easy to get that

$$a^{\Delta\Delta}(t) = \frac{At^\alpha(q^\alpha - 1)(q^\alpha - q)}{t^2 q(q-1)^2} \leq 0.$$

Repeated applications of Theorem 2.2, gives us that

$$x^{\Delta\Delta}(t) \pm Bt^\beta b_0 \frac{(-1)^n}{t^2} x(qt) = 0$$

is oscillatory, for all $\beta > 0$, $B > 0$. So the equation

$$x^{\Delta\Delta}(t) \pm Bb_0 \frac{(-1)^n}{t^{2-\beta}} x(qt) = 0$$

is oscillatory, for all $\beta > 0$, $B > 0$. This means that the equation

$$x^{\Delta\Delta}(t) + b\frac{(-1)^n}{t^c} x(qt) = 0$$

is oscillatory, for $b \neq 0$, $c < 2$. □

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