Asymptotic properties of solutions of a $2n^{th}$ order differential equation on a time scale

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ABSTRACT: In this paper we are concerned with a 2n-th order linear self-adjoint differential equation on a time scale. The results generalize known results for the corresponding ordinary differential equations and for difference equations. We define type I and type II solutions, prove the existence of these solutions, and verify asymptotic properties of these solutions. A quadratic functional corresponding to the differential equation on a time scale is defined and is used to prove several of the results in this paper.
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1 INTRODUCTION

In this paper we will be concerned with the $2n$th-order differential equation

$$Lx(t) = \left[p(t)x^{\Delta^n}(t)\right]^{\Delta^n} + q(t)x(\sigma^n(t)) = 0 \tag{1}$$

for $t \in \mathbb{T}$. We assume $\mathbb{T}$ is a closed subset of the real numbers $\mathbb{R}$ that is unbounded above. We call $\mathbb{T}$ a time scale.

Definition For $t \in \mathbb{T}$ define the forward jump operator σ by

$$\sigma(t) := \inf\{\tau \in \mathbb{T} : \tau > t\}.$$

We define the backwards jump operator ρ for $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, by

$$\rho(t) := \sup\{\tau \in \mathbb{T} : \tau < t\}.$$

If $\sigma(t) = t$ we say t is *right-dense*, and if $\sigma(t) > t$ we say t is *right-scattered*. Similarly, if $\rho(t) = t$ then t is *left-dense*, and if $\rho(t) < t$ then t is *left-scattered*. We assume $\sigma^0(t) = t$, and for any integer $n > 0$, we define $\sigma^n : \mathbb{T} \to \mathbb{R}$ by

$$\sigma^n(t) := \sigma(\sigma^{n-1}(t)).$$

We assume throughout that the time scale $\mathbb{T}$ has the topology it inherits from the standard topology on $\mathbb{R}$. We also assume $p, q : \mathbb{T} \to \mathbb{R}$ are continuous and

$$p(t) > 0$$

on $\mathbb{T}$.

Definition Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$; define $x^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\Big|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]\Big| < \epsilon\, |\sigma(t) - s|$$

for all $s \in U$. We call $x^\Delta(t)$ the *delta derivative* of x at t. Note if $\mathbb{T} = \mathbb{Z}$, then

$$x^\Delta(t) = \Delta x(t) := x(t+1) - x(t).$$

Moreover, if $\mathbb{T} = \mathbb{R}$, then

$$x^\Delta(t) = x'(t).$$

Hence, our results contain differential equations and difference equations as special cases. Finally, if $h > 0$ and $\mathbb{T} = h\mathbb{Z} = \{hn : n \in \mathbb{Z}\}$, then

$$x^\Delta(t) = \frac{x(t+h) - x(t)}{h}.$$

Definition For $n \geq 1$, define

$$x^{\Delta^n}(t) := \Big\{x^\Delta(t)\Big\}^{\Delta^{n-1}};$$

assume $x^{\Delta^0}(t) = x(t)$.

Definition We say $x : \mathbb{T} \to \mathbb{R}$ is *right-dense continuous* provided for any $t \in \mathbb{T}$ such that $\rho(t) = t$, the lefthand limit of x at t exists (and is finite). Also, if $t \in \mathbb{T}$ and $\sigma(t) = t$, then

$$\lim_{\tau \to t^+} x(\tau) = x(t).$$

Note if $x : \mathbb{T} \to \mathbb{R}$ is continuous, then $x(\sigma(t))$ is right-dense continuous on $\mathbb{T}$.

Definition We define

$$\begin{aligned} D \;:=\; & \Big\{x : \mathbb{T} \to \mathbb{R} :\; x^{\Delta^k}(\sigma^i(t)) \text{ is delta differentiable on } \mathbb{T} \text{ for } \; 0 \leq k \leq n-1 \\ & \text{and } 0 \leq i \leq n; \;\; \Big[p(\sigma^i(t))x^{\Delta^n}(\sigma^i(t))\Big]^{\Delta^k} \text{ is delta differentiable on } \mathbb{T} \text{ for} \\ & 0 \leq i, k \leq n-1; \;\text{ and }\; [p(t)x^{\Delta^n}(t)]^{\Delta^n} \text{ is right-dense continuous on } \mathbb{T}\Big\}. \end{aligned}$$

We say x is a solution of $Lx = 0$ on $\mathbb{T}$ provided $x \in D$ and $Lx(t) = 0$ for all $t \in \mathbb{T}$. Our results will hold only for those time scales $\mathbb{T}$ for which D is nonempty. Moreover, we need to assume below that the time scale $\mathbb{T}$ is such that

$$[x^{\Delta^{n-1}}(\sigma^n(t))]^\Delta = x^{\Delta^n}(\sigma^n(t)).$$

This is true for time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = h\mathbb{Z}$ for any $h > 0$, but of course is not true in general.

Definition For $x \in D$, set

$$\begin{aligned} Fx(t) \; := \; & x^{\Delta^{n-1}}(\sigma^{n-1}(t))p(\sigma^{n-1}(t))x^{\Delta^n}(\sigma^{n-1}(t)) \\ & -(-1)^n \sum_{i=0}^{n-2}(-1)^i x^{\Delta^i}(\sigma^n(t)) \left[p(\sigma^i(t))x^{\Delta^n}(\sigma^i(t))\right]^{\Delta^{n-1-i}}; \end{aligned} \quad (2)$$

this quadratic functional F will be important in the study of the asymptotic properties of solutions of $Lx(t) = 0$. Note that $F[kx(t)] = k^2 F(x(t))$.

In the proof of the following lemma, we will make repeated use of the product formula

$$[f(t)g(t)]^\Delta = f(\sigma(t))g^\Delta(t) + f^\Delta(t)g(t),$$

where f and g are interchangeable.

Lemma 1 *If x is a solution of* (1) *on $\mathbb{T}$, then*

$$[Fx(t)]^\Delta = p(\sigma^{n-1}(t)) \left[x^{\Delta^n}(\sigma^{n-1}(t))\right]^2 + (-1)^n q(t) x^2(\sigma^n(t)) \quad (3)$$

for $t \in \mathbb{T}$. In particular, if

$$(-1)^n q(t) \geq 0 \quad (4)$$

on $\mathbb{T}$, then F is nondecreasing along solutions x of (1) *for $t \in \mathbb{T}$.*

Proof: Assume x is a solution of (1). Then

$$\begin{aligned} [Fx(t)]^\Delta \; = \; & x^{\Delta^{n-1}}(\sigma^n(t)) \left[p(\sigma^{n-1}(t))x^{\Delta^n}(\sigma^{n-1}(t))\right]^\Delta & (5) \\ & + x^{\Delta^n}(\sigma^{n-1}(t))p(\sigma^{n-1}(t))x^{\Delta^n}(\sigma^{n-1}(t)) & (6) \\ & -(-1)^n \sum_{i=0}^{n-2}(-1)^i x^{\Delta^i}(\sigma^n(t)) \left[p(\sigma^i(t))x^{\Delta^n}(\sigma^i(t))\right]^{\Delta^{n-i}} & (7) \\ & -(-1)^n \sum_{i=0}^{n-2}(-1)^i x^{\Delta^{i+1}}(\sigma^n(t)) \left[p(\sigma^{i+1}(t))x^{\Delta^n}(\sigma^{i+1}(t))\right]^{\Delta^{n-i-1}} & (8) \end{aligned}$$

where (5) and (6) result from the product rule of the first advanced times the delta derivative of the second plus the second times the derivative of the first, while (7) and (8) come from the product rule of the first times the delta derivative of the second plus the derivative of the first times the second advanced. Next, evaluating the sum in (8) at $n-2$ and reindexing, we obtain

$$\begin{aligned}
[Fx(t)]^{\Delta} &= x^{\Delta^{n-1}}(\sigma^n(t))\left[p(\sigma^{n-1}(t))x^{\Delta^n}(\sigma^{n-1}(t))\right]^{\Delta} + p(\sigma^{n-1}(t))\left[x^{\Delta^n}(\sigma^{n-1}(t))\right]^2 \\
&\quad -(-1)^n(-1)^{n-2}x^{\Delta^{n-1}}(\sigma^n(t))\left[p(\sigma^{n-1}(t))x^{\Delta^n}(\sigma^{n-1}(t))\right]^{\Delta} \\
&\quad -(-1)^n\sum_{i=0}^{n-2}(-1)^i x^{\Delta^i}(\sigma^n(t))\left[p(\sigma^i(t))x^{\Delta^n}(\sigma^i(t))\right]^{\Delta^{n-i}} \\
&\quad -(-1)^n\sum_{i=1}^{n-2}(-1)^{i-1} x^{\Delta^i}(\sigma^n(t))\left[p(\sigma^i(t))x^{\Delta^n}(\sigma^i(t))\right]^{\Delta^{n-i}} \\
&= p(\sigma^{n-1}(t))\left[x^{\Delta^n}(\sigma^{n-1}(t))\right]^2 - (-1)^n x(\sigma^n(t))\left[p(t)x^{\Delta^n}(t)\right]^{\Delta^n} \\
&= p(\sigma^{n-1}(t))\left[x^{\Delta^n}(\sigma^{n-1}(t))\right]^2 + (-1)^n q(t)x^2(\sigma^n(t)),
\end{aligned}$$

since x is a solution of (1).

Definitions If $x(t)$ is a solution of (1) such that $Fx(t) \le 0$ in a neighborhood of ∞, we say $x(t)$ is a *type I solution* of (1). If $x(t)$ is a solution of (1) such that $Fx(t) < 0$ for $t \in \mathbb{T}$, then we say $x(t)$ is a *strict type I solution.* If x solves (1) and $Fx(t) > 0$ near ∞, then x is a *type II solution.* In view of Lemma 1, all solutions of (1) are type I or type II solutions if (4) holds.

Hartman [3] defined generalized zeros of solutions of n^{th}-order linear difference equations. We now would like to define what we mean by a *generalized zero of order* m, $0 \le m \le 2n-1$, for a nontrivial solution $x(t)$ of (1) at $t_0 \in \mathbb{T}$. If $t_0 = \min \mathbb{T}$, then we say $x(t)$ has a generalized zero of order m at t_0 provided

$$x^{\Delta^k}(t_0) = 0,$$

$0 \le k \le m-1$. If $t_0 > \min \mathbb{T}$, then x has a generalized zero of order m at t_0 provided if t_0 is left-dense then

$$x^{\Delta^k}(t_0) = 0,$$

$0 \le k \le m-1$; if t_0 is left-scattered, then

$$x(\rho(t_0)) \ne 0 \ \text{ and } \ x^{\Delta^k}(t_0) = 0$$

for $0 \le k \le m-1$, or t_0 is right-scattered with

$$x^{\Delta^k}(t_0) = 0$$

for $0 \le k \le m-2$ and

$$(-1)^m x(\rho(t_0))x(\sigma^{m-1}(t_0)) > 0.$$

(This last case happens only if $\rho(t_0) < t_0 < \sigma(t_0) < \cdots < \sigma^{m-1}(t_0)$.)

2 MAIN RESULTS

Theorem 2 *Assume* (4) *holds on* $\mathbb{T}$. *Any nontrivial solution* x *of* (1) *with*

$$x^{\Delta^k}(\sigma^n(t_0)) = 0,$$

$0 \leq k \leq n-1$, *for some* $t_0 \in \mathbb{T}$ *is a type* II *solution. In particular,* (1) *has at least* n *linearly independent type* II *solutions.*

Proof: Let $t_0 \in \mathbb{T}$, and let $x_j(t)$, $0 \leq j \leq n-1$, be solutions of $Lx = 0$ satisfying

$$x_j^{\Delta^k}(\sigma^n(t_0)) = 0, \quad 0 \leq k \leq n-1 \tag{9}$$

$$\left[p(t)x_j^{\Delta^k}(t)\right]^{\Delta^k}(t_0) = \delta_{jk}, \quad 0 \leq j,k \leq n-1.$$

It is easy to see that $\{x_j(t)\}_{j=0}^{n-1}$ is a set of n linearly independent solutions of (1), and because of (9),

$$Fx_j(\sigma(t_0)) = 0. \tag{10}$$

Integrating both sides of (3) from $\sigma(t_0)$ to t and using (10) we get

$$Fx_j(t) = \int_{\sigma(t_0)}^{t} \left[p(\sigma^{n-1}(s))\left[x_j^{\Delta^n}(\sigma^{n-1}(s))\right]^2 + (-1)^n q(s) x_j^2(\sigma^n(s))\right]\Delta s.$$

Since $x_j(t)$ is a nontrivial solution, by Lemma 1 we can pick $\tau > \sigma(t_0)$ sufficiently large so that

$$Fx_j(t) > 0$$

for all $t \geq \tau$. Hence, $x_1, \ldots, x_n$ are n linearly independent type II solutions.

Theorem 3 *Assume* (4) *holds. Then* (1) *has* n *linearly independent type* I *solutions.*

Proof: Let $a \in \mathbb{T}$. For each fixed $s \in \mathbb{T}$, $s > \sigma^{2n}(a)$, we let $v_k(t,s)$ be a nontrivial solution of (1) for $1 \leq k \leq n$, satisfying the $2n-1$ boundary conditions

$$\begin{array}{ll} v_k^{\Delta^i}(\sigma^n(a),s) = 0 & \text{for } 0 \leq i \leq n-1 \text{ but } i \neq k-1 \\ v_k^{\Delta^i}(\sigma^n(s),s) = 0 & \text{for } 0 \leq i \leq n-1. \end{array} \tag{11}$$

Then define

$$u_k(t,s) := \frac{v_k(t,s)}{\sqrt{[v_k(\sigma^n(a),s)]^2 + [v_k^{\Delta}(\sigma^n(a),s)]^2 + \cdots + \left[v_k^{\Delta^{2n-1}}(\sigma^n(a),s)\right]^2}} \tag{12}$$

for $1 \leq k \leq n$ and $s > \sigma^{2n}(a)$. Then $u_k(t,s)$ is a solution of (1) satisfying

$$\sum_{i=0}^{2n-1}\left[u_k^{\Delta^i}(\sigma^n(a),s)\right]^2 = 1.$$

Thus, for each k, and $s \in \mathbb{T}$ with $s > \sigma^{2n}(a)$ (recall $\mathbb{T}$ is unbounded above), from the set of points

$$\left\{(u_k(\sigma^n(a),s), u_k^{\Delta}(\sigma^n(a),s), \ldots, u_k^{\Delta^{2n-1}}(\sigma^n(a),s))\right\}_{s\in\mathbb{T}}$$

in $\mathbb{R}^{2n}$ we can pick a convergent sequence

$$\left\{(u_k(\sigma^n(a), s_{j_k}), u_k^{\Delta}(\sigma^n(a), s_{j_k}), \ldots, u_k^{\Delta^{2n-1}}(\sigma^n(a), s_{j_k}))\right\}_{j=1}^{\infty}.$$

Let

$$v_{i+1,k} := \lim_{j\to\infty} u_k^{\Delta^i}(\sigma^n(a), s_{j_k}) \tag{13}$$

for $0 \le i \le 2n-1$. Then

$$\sum_{i=0}^{2n-1} v_{i+1,k}^2 = 1. \tag{14}$$

Further, let y_k, $1 \le k \le n$, be the solutions of (1) satisfying

$$y_k^{\Delta^i}(\sigma^n(a)) = v_{i+1,k} \tag{15}$$

for $0 \le i \le 2n-1$, $1 \le k \le n$; note that the y_k are nontrivial solutions by (14) and (15). Since $v_k(t,s)$ satisfies (11), formula (12) implies $u_k(t,s)$, $1 \le k \le n$, also satisfies (11). Hence, we have

$$Fu_k(\sigma(s_{j_k}), s_{j_k}) = 0;$$

as $Fu_k(t, s_{j_k})$ is nondecreasing by Lemma 1,

$$Fu_k(t, s_{j_k}) \le 0$$

for $t \in \mathbb{T}$ with $\sigma^n(a) \le t \le \sigma(s_{j_k})$, for $1 \le k \le n$. So, for each $t_1 \in \mathbb{T}$ with $t_1 > \sigma^n(a)$, there exists j_{t_1} such that $\sigma(s_{j_k}) > t_1$ for all $j_k \ge j_{t_1}$. Then

$$Fu_k(t_1, s_{j_k}) \le 0$$

for all $j_k \ge j_{t_1}$. Taking the limit as $j \to \infty$, we get that

$$Fy_k(t_1) \le 0$$

for $1 \le k \le n$. As $t_1 > \sigma^n(a)$ was arbitrary,

$$Fy_k(t) \le 0$$

for all $t > \sigma^n(a)$, for $1 \le k \le n$. It follows that the y_k are type I solutions of (1) for $1 \le k \le n$. By (11),(12),(13), and (15) we have that

$$y_k^{\Delta^i}(\sigma^n(a)) = 0 \;\text{ for }\; 0 \le i \le n-1 \;\text{ if }\; i \ne k-1.$$

If $y_k^{\Delta^{k-1}}(\sigma^n(a)) = 0$, then y_k would have a zero of order n at $\sigma^n(a)$, so that y_k would be a type II solution by Theorem 2. Consequently, $y_k^{\Delta^{k-1}}(\sigma^n(a)) \ne 0$ for $1 \le k \le n$, and the y_k are linearly independent.

Theorem 4 *If* (4) *holds on* $\mathbb{T}$ *and* x *is a type I solution of* (1)*, then*

$$(-1)^n \int^{\infty} q(t) x^2(\sigma^n(t)) \Delta t < \infty, \tag{16}$$

and

$$\int^{\infty} p(t) \left[x^{\Delta^n}(t)\right]^2 \Delta t < \infty. \tag{17}$$

If $q(t) \ne 0$ *in a neighborhood of* ∞*, then every nontrivial type I solution of* (1) *is a strict type I solution.*

Proof: Let x be a type I solution of (1) on $\mathbb{T}$; then $Fx(t) \leq 0$ for $t \in \mathbb{T}$. Let

$$M := \lim_{t \to \infty} Fx(t) \leq 0.$$

Integrating both sides of (3) from $t_0 \in \mathbb{T}$ to ∞ yields

$$\begin{aligned} M - Fx(t_0) &= \int_{t_0}^{\infty} p(\sigma^{n-1}(t)) \left[x^{\Delta^n}(\sigma^{n-1}(t))\right]^2 \Delta t \\ &+ \int_{t_0}^{\infty} (-1)^n q(t) x^2(\sigma^n(t)) \Delta t. \end{aligned}$$

Thus (16) and (17) hold. Now assume $q(t) \neq 0$ in a neighborhood of ∞, and that v is a nontrivial type I solution of (1); then $Fv(t) \leq 0$ for $t \in \mathbb{T}$. Suppose there exists a $t_0 \in \mathbb{T}$ such that $Fv(t_0) = 0$. Then $Fv(t) \equiv 0$ on $[t_0, \infty) \cap \mathbb{T}$, by Lemma 1. But then $[Fv(t)]^\Delta = 0$ on $[t_0, \infty) \cap \mathbb{T}$, so that from (3) we have

$$p(\sigma^{n-1}(t)) \left[v^{\Delta^n}(\sigma^{n-1}(t))\right]^2 + (-1)^n q(t) v^2(\sigma^n(t)) \equiv 0$$

for $t \geq t_0 \in \mathbb{T}$. Since (4) holds all terms are nonnegative; moreover, $q(t) \neq 0$ near ∞ gives that v is the trivial solution, contrary to assumption. Therefore, $Fv(t) < 0$ for all $t \in \mathbb{T}$, and v is a strict type I solution of (1).

We conjecture that if (4) holds and $\liminf_{t \to \infty} (-1)^n q(t) > 0$, then (1) has n linearly independent type I solutions v_k satisfying $\lim_{t \to \infty} v_k(t) = 0$ for $1 \leq k \leq n$.

Theorem 5 *If* (4) *holds on* $\mathbb{T}$, *then no nontrivial solution of* (1) *can have a zero of order* n *at* $\sigma^n(t_1)$ *followed by a generalized zero of order* n *at* $\sigma^n(t_2)$, *where* $t_1, t_2 \in \mathbb{T}$ *with* $t_2 > \sigma^n(t_1)$ *and* $\sigma^n(t_2)$ *is left-scattered.*

Proof: Assume (4) holds. Assume x is a nontrivial solution of (1) and $t_1 \in \mathbb{T}$ such that

$$x^{\Delta^i}(\sigma^n(t_1)) = 0$$

for $0 \leq i \leq n-1$. Then $Fx(\sigma(t_1)) = 0$; by Lemma 1,

$$Fx(t) \geq 0 \tag{18}$$

for all $t \geq \sigma(t_1) \in \mathbb{T}$. Suppose there is a $t_2 > \sigma^n(t_1) \in \mathbb{T}$ such that x has a generalized zero of order n at $\sigma^n(t_2)$. Then $x(\sigma^{n-1}(t_2)) \neq 0$, and by (2),

$$\begin{aligned} Fx(t_2) &= x^{\Delta^{n-1}}(\sigma^{n-1}(t_2)) p(\sigma^{n-1}(t_2)) x^{\Delta^n}(\sigma^{n-1}(t_2)) \\ &= \frac{(-1)^{n-1} x(\sigma^{n-1}(t_2)) p(\sigma^{n-1}(t_2)) x^{\Delta^n}(\sigma^{n-1}(t_2))}{\alpha^{n-1}(t_2)} \\ &= \frac{(-1)^{n-1} x(\sigma^{n-1}(t_2)) p(\sigma^{n-1}(t_2))}{\alpha^{n-1}(t_2)} \left[\frac{x(\sigma^{2n-1}(t_2))}{\beta(t_2)} + \frac{(-1)^n x(\sigma^{n-1}(t_2))}{\alpha^n(t_2)}\right] \\ &= \frac{(-1)^{n-1} x(\sigma^{n-1}(t_2)) x(\sigma^{2n-1}(t_2)) p(\sigma^{n-1}(t_2))}{\alpha^{n-1}(t_2)\beta(t_2)} - \frac{x^2(\sigma^{n-1}(t_2)) p(\sigma^{n-1}(t_2))}{\alpha^{2n-1}(t_2)}, \end{aligned}$$

where

$$\alpha(t) := \sigma^n(t) - \sigma^{n-1}(t)$$

and

$$\beta(t) := [\sigma^{2n-1}(t) - \sigma^{2n-2}(t)][\sigma^{2n-2}(t) - \sigma^{2n-3}(t)] \cdots [\sigma^{n}(t) - \sigma^{n-1}(t)];$$

we assume $\alpha(t_2)$ and $\beta(t_2)$ are nonzero when $x(\sigma^{2n-1}(t_2)) \neq 0$. Since x has a generalized zero of order n at $\sigma^n(t_2)$,

$$(-1)^n x(\sigma^{n-1}(t_2)) x(\sigma^{2n-1}(t_2)) \geq 0.$$

Therefore $Fx(t_2) < 0$, a contradiction of (18) since $t_2 > \sigma(t_1)$.

Theorem 6 *Every unbounded solution near ∞ of* (1)*, where*

$$\liminf_{t\to\infty} q(t) > 0 \tag{19}$$

and

$$0 < \liminf_{t\to\infty} p(t) < \infty \tag{20}$$

for $t \in \mathbb{T}$, is oscillatory.

Proof: Let x be an unbounded solution of (1). Assume that x is nonoscillatory; then without loss of generality there exists t_0 in $\mathbb{T}$ such that $x(t) > 0$ for all $t \geq t_0$ in $\mathbb{T}$. Since x is a solution of (1), by repeated application of Rolle's Theorem we can assume x is increasing. From (19) we can also assume t_0 is sufficiently large so that $q(t) > 0$ for $t \geq t_0$ in $\mathbb{T}$. Therefore

$$\left[p(t)x^{\Delta^n}(t)\right]^{\Delta^n} = -q(t)x(\sigma^n(t)) < 0 \tag{21}$$

for $t \geq t_0$ in $\mathbb{T}$, and

$$\lim_{t\to\infty} \left[p(t)x^{\Delta^n}(t)\right]^{\Delta^n} = \lim_{t\to\infty} -q(t)x(\sigma^n(t)) = -\infty. \tag{22}$$

And yet

$$\left[p(t)x^{\Delta^n}(t)\right]^{\Delta^{n-1}} - \left[p(t_0)x^{\Delta^n}(t_0)\right]^{\Delta^{n-1}} = \int_{t_0}^{t} \left[p(s)x^{\Delta^n}(s)\right]^{\Delta^n} \Delta s \to -\infty. \tag{23}$$

by (22). As a result, there is a $t_1 > t_0$ in $\mathbb{T}$ such that

$$\left[p(t)x^{\Delta^n}(t)\right]^{\Delta^{n-1}} < 0$$

for all $t \geq t_1$ in $\mathbb{T}$,

$$\liminf_{t\to\infty} \left[p(t)x^{\Delta^n}(t)\right]^{\Delta^{n-1}} = -\infty \tag{24}$$

for $t \in \mathbb{T}$ using (21), (22), (23), and so on. By continuing this process of integrating each successive expression we obtain

$$\liminf_{t\to\infty} \left[p(t)x^{\Delta^n}(t)\right]^{\Delta^i} = -\infty$$

for $t \in \mathbb{T}$, $i = n-2, n-3, \ldots, 1, 0$. Using (20) and similar reasoning, we see for $i = n, n-1, \ldots, 1, 0$ that

$$\liminf_{t\to\infty} x^{\Delta^i}(t) = -\infty$$

as well. Consequently

$$\liminf_{t\to\infty} x(t) = -\infty,$$

a contradiction of our initial assumption that $x(t) > 0$ for $t \geq t_0$ in $\mathbb{T}$. Thus every unbounded solution of (1) is oscillatory under conditions (19) and (20).

Example 7 *The differential equation*

$$x^{\Delta\Delta\Delta\Delta}(t) + x(\sigma^2(t)) = 0,$$

for $t \in \mathbb{T}$, satisfies the hypotheses of most of the theorems in this paper. Note that if $\mathbb{T} = \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of nonnegative real numbers, then

$$\begin{aligned} x_1(t) &= e^{\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right), \\ x_2(t) &= e^{\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right), \\ x_3(t) &= e^{-\frac{\sqrt{2}}{2}t}\cos\left(\frac{\sqrt{2}}{2}t\right), \\ x_4(t) &= e^{-\frac{\sqrt{2}}{2}t}\sin\left(\frac{\sqrt{2}}{2}t\right), \end{aligned}$$

are solutions on $\mathbb{R}^+$. Note that $x_1(t)$ and $x_2(t)$ are unbounded near ∞ and as guaranteed by Theorem 6 are oscillatory near ∞. It can be shown that

$$Fx_1(t) = Fx_2(t) = \frac{1}{\sqrt{2}}e^{\frac{t}{\sqrt{2}}},$$

and it follows that $x_1(t)$ and $x_2(t)$ are type II solutions. Also it can be shown that

$$Fx_3(t) = Fx_4(t) = -\frac{1}{\sqrt{2}}e^{-\frac{t}{\sqrt{2}}},$$

so $x_3(t)$ and $x_4(t)$ are type I solutions. Hence, the conjecture noted after Theorem 4 holds.

Note that Peil and Peterson [6] studied the asymptotic behavior of solutions of (1) with $\mathbb{T} \equiv \mathbb{Z}$. For fourth order difference equations see Smith and Taylor [8]. For fourth order ordinary differential equations see Jones [4] and Svec [9]. For results of the type in this paper for third order difference equations see Peterson [7]. For results for third order differential equations on a time scale see Morelli and Peterson [5].

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