OSCILLATION FOR NONLINEAR SECOND ORDER DYNAMIC EQUATIONS ON A TIME SCALE

MARTIN BOHNER, LYNN ERBE[1], AND ALLAN PETERSON[1]

Abstract. We obtain some oscillation criteria for solutions to the nonlinear dynamic equation

$$x^{\Delta\Delta} + q(t)x^{\Delta^\sigma} + p(t)(f \circ x^\sigma) = 0,$$

on time scales. In particular, no explicit sign assumptions are made with respect to the coefficients $p(t)$, $q(t)$. We illustrate the results by several examples, including a superlinear Emden–Fowler dynamic equation.

1. Introduction

Consider the second order nonlinear dynamic equation

$$x^{\Delta\Delta} + q(t)x^{\Delta^\sigma} + p(t)(f \circ x^\sigma) = 0, \tag{1.1}$$

where p and q are real–valued, right–dense continuous functions on a time scale $\mathbb{T} \subset \mathbb{R}$, with $\sup \mathbb{T} = \infty$. We also assume throughout that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$f'(x) > 0 \quad \text{and} \quad xf(x) > 0 \quad \text{for} \quad x \neq 0. \tag{1.2}$$

In contrast to most results dealing with second order nonlinear oscillation, we do not make any explicit sign assumptions on p and q.

For completeness, we recall the following concepts related to the notion of time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ and, since oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. The *forward* and *backward jump operators* are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}, \quad \rho(t) := \sup\{s \in \mathbb{T}, s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be *left–dense* if $\rho(t) = t$, *right–dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left–scattered* if $\rho(t) < t$ and *right–scattered* if $\sigma(t) > t$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right–dense continuous* (rd–continuous) provided g is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of

2000 *Mathematics Subject Classification.* 39A10.

[1]Research was supported by NSF Grant 0072505.

all such rd–continuous functions is denoted by $C_{rd}(\mathbb{T})$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

The assumption (1.2) allows f to be of superlinear or sublinear growth, say

$$f(x) = x^\gamma, \quad \gamma > 0 \ \text{(quotient of odd positive integers).} \tag{1.3}$$

In several papers ([4], [13]), (1.1) has been studied with $p > 0$ and assuming the nonlinearity has the property

$$\left|\frac{f(x)}{x}\right| \geq K \quad \text{for} \quad x \neq 0. \tag{1.4}$$

This essentially says that the equation is, in some sense, not too far from being linear. In the papers [10] and [11] (see also [8]) it was shown that one may relate oscillation and boundedness of solutions of the nonlinear equation (1.1) to a related linear equation, which in the case $q(t) \equiv 0$ reduces to

$$x^{\Delta\Delta} + \lambda p(t) x^\sigma = 0, \tag{1.5}$$

where $\lambda > 0$, for which many oscillation criteria are known (see e.g. [3], [4], [7], and [12]). In particular, analogues of the results due to Erbe [6] and others for the continuous case $\mathbb{T} = \mathbb{R}$ were extended. However, in the papers [10] and [11] it was assumed that the nonlinearity has the property

$$f'(x) \geq \frac{f(x)}{x} \quad \text{for} \quad x \neq 0. \tag{1.6}$$

We shall show by means of an example that this condition can be relaxed.
Throughout this paper, we shall restrict attention to solutions of (1.1) which exist on some interval of the form $[T_x, \infty)$, where $T_x \in \mathbb{T}$ may depend on the particular solution. In Section 2 we present some preliminary results on the chain rule, integration by parts, and an auxiliary lemma. Section 3 contains the main results on oscillation and several examples are given in Section 4 as well as a comparison with some previous results.

2. Preliminary Results

On an arbitrary time scale $\mathbb{T}$, the usual chain rule from calculus is no longer valid (see Bohner and Peterson [3], pp 31). One form of the extended chain rule, due to S. Keller [15] and generalized to measure chains by C. Pötzsche [17], is as follows. (See also Bohner and Peterson [3], pp 32.)

Lemma 2.1. *Assume $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}$. Assume further that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable* *and satisfies*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t). \tag{2.1}$$

We shall also need the following integration by parts formula (cf. [3, Theorem 1.77]), which is a simple consequence of the product rule and which we formulate as follows:

Lemma 2.2. *Let $a, b \in \mathbb{T}$ and assume $f^{\Delta}, g^{\Delta} \in C_{rd}$. Then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^{\Delta}(t)g(t)\Delta t. \tag{2.2}$$

We also introduce the following condition:

We say that a function $g : \mathbb{T} \to \mathbb{R}$ satisfies **condition (A)** if the following condition holds:

$$\liminf_{t\to\infty} \int_T^t g(s)\Delta s \geq 0 \quad \text{and} \ \not\equiv 0 \tag{2.3}$$

for all large T. It can be shown that (2.3) implies either $\int_a^{\infty} g(s)\Delta s = +\infty$ or that

$$\int_T^{\infty} g(s)\Delta s = \lim_{t\to\infty} \int_T^t g(s)\Delta s$$

exists and satisfies $\int_T^{\infty} g(s)\Delta s \geq 0$ for all large T.

We state the following lemma which gives another simple consequence of condition (A).

Lemma 2.3. *Suppose that g satisfies condition (A). Then given any T_0 there exists $T_1 \geq T_0$ so that*

$$\int_{T_1}^t g(s)\Delta s \geq 0 \quad \textit{for all} \quad t \geq T_1. \tag{2.4}$$

Proof. Indeed, if no such $T_1 \geq T_0$ exists, then for any $T > T_0$ fixed but arbitrary, we define

$$T_1 = T_1(T) := \sup\{t > T : \int_T^t q(s)\Delta s < 0\}.$$

If $T_1 = \infty$, then choosing $t_n \to \infty$ such that $\int_T^{t_n} q(s)\Delta s < 0$ for all n, we obtain a contradiction to (2.3). Hence, we must have $T_1 < \infty$ which implies $\int_{T_1}^t q(s)\Delta s \geq 0$ for all $t \geq T_1$. □

3. Main Results

We recall that a solution of equation (1.1) is said to be **oscillatory** on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be **nonoscillatory**. Equation (1.1) is said to be oscillatory in case all of its solutions are oscillatory. We have the following lemma which describes the behavior of a nonoscillatory solution of (1.1) for the case when (2.3) holds. In the statement of the lemma we let the function $r(t)$ (see [3]) be given in terms of the generalized exponential function by $r(t) := e_q(t, t_0)$. If $q \in \mathcal{R}^+$ (i.e., q is positively regressive), it follows that $r(t) > 0$ for all $t \geq t_0$ and furthermore, may be characterized as the unique solution of the IVP $x^\Delta = q(t)x$, $x(t_0) = 1$.)

Note that after multiplying by $r(t)$, (1.1) may be written as

$$(r(t)x^\Delta)^\Delta + p_1(t)(f \circ x^\sigma) = 0, \tag{3.1}$$

where $p_1(t) := r(t)p(t)$.

Lemma 3.1. *Let x be a nonoscillatory solution of* (1.1) *and assume that $p_1(t)$ satisfies condition (A)* , (1.2) *holds, $q \in \mathcal{R}^+$, and*

$$\int_T^\infty \frac{1}{r(t)} \Delta t = \infty.$$

Then there exists $T_1 \geq T$ such that

$$x(t)x^\Delta(t) > 0 \quad \textit{for} \quad t \geq T_1.$$

Proof. Suppose that x is a nonoscillatory solution of (1.1) and without loss of generality, assume $x(t) > 0$ for $t \geq T_0$. Since $p_1(t)$ satisfies condition (A), we may assume by Lemma 2.3 that $T_1 \geq T_0$ is sufficiently large so that

$$\int_{T_1}^t p_1(s)\Delta s \geq 0 \quad \text{for all} \quad t \geq T_1. \tag{3.2}$$

Let us assume, for the sake of contradiction, that $x^\Delta(t)$ is not strictly positive for all large t. First consider the case when $x^\Delta(t) < 0$ for all large t. Then without loss of generality $x^\Delta(t) < 0$ for all $t \geq T_1 \geq T_0$. An integration of (3.1) for $t > T_1$ gives

$$r(t)x^\Delta(t) + \int_{T_1}^t p_1(s) f(x^\sigma(s))\Delta s = r(T_1)x^\Delta(T_1) < 0. \tag{3.3}$$

Now by the integration by parts formula (2.2) we have

$$\begin{aligned} \int_{T_1}^t p_1(s) f(x^\sigma(s))\Delta s &= f(x(t)) \int_{T_1}^t p_1(s)\Delta s \\ &\quad - \int_{T_1}^t (f \circ x)^\Delta(s) \int_{T_1}^s p_1(u)\Delta u \Delta s. \end{aligned} \tag{3.4}$$

By the chain rule (2.1) we have (with $g(t) = x(t)$)

$$(f \circ x)^{\Delta}(t) = \left\{ \int_0^1 f'(x(t) + h\mu(t)x^{\Delta}(t))dh \right\} x^{\Delta}(t) \le 0,$$

since $f'(u) > 0$ for all $u \neq 0$ and $x^{\Delta}(t) < 0$. Hence, it follows that

$$\int_{T_1}^{t} (f \circ x)^{\Delta}(s) \int_{T_1}^{s} p_1(u)\Delta u \Delta s \le 0 \tag{3.5}$$

and so from (3.4) we have

$$\int_{T_1}^{t} p_1(s) f(x^{\sigma}(s))\Delta s \ge f(x(t)) \int_{T_1}^{t} p_1(s)\Delta s \ge 0. \tag{3.6}$$

Consequently, from (3.3) we have

$$r(t)x^{\Delta}(t) \le r(T_1)x^{\Delta}(T_1) < 0, \quad t \ge T_1, \tag{3.7}$$

and now dividing by $r(t)$ and integrating (3.7) yields

$$x(t) \le x(T_1) + r(T_1)x^{\Delta}(T_1) \int_{T_1}^{t} \frac{1}{r(s)} \Delta s \to -\infty \tag{3.8}$$

which is a contradiction. Hence $x^{\Delta}(t)$ is not negative for all large t and since we are assuming $x^{\Delta}(t)$ is not positive for all large t, it follows that $x^{\Delta}(t)$ must change sign infinitely often.

Make the "Riccati–like" substitution

$$w(t) := -\frac{r(t)x^{\Delta}(t)}{f(x(t))}, \quad t \ge T_0. \tag{3.9}$$

We may suppose that $T_1 > T_0$ is sufficiently large so that

$$\liminf_{t\to\infty} \int_{T_1}^{t} p_1(s)\Delta s \ge 0$$

holds and is such that $w(T_1) > 0$ (i.e., $x^{\Delta}(T_1) < 0$).

Differentiating w and using the chain rule (2.1) gives

$$\begin{aligned} w^{\Delta}(t) &= p_1(t) + w^2(t) \frac{f(x(t))}{r(t) f(x^{\sigma}(t))} \left\{ \int_0^1 f'(x(t) + h\mu(t)x^{\Delta}(t))dh \right\} \\ &\ge p_1(t), \quad t \ge T_1, \end{aligned}$$

and this yields

$$w(t) \ge w(T_1) + \int_{T_1}^{t} p_1(s)\Delta s, \quad t \ge T_1. \tag{3.10}$$

Now taking the lim inf of both sides of (3.10) we have since $p_1(t)$ satisfies condition (A), that

$$\liminf_{t\to\infty} w(t) \ge w(T_1) > 0,$$

which implies that $x^{\Delta}(t) < 0$ for all large t, which is a contradiction to the assumption that $x^{\Delta}(t)$ changes sign infinitely often. □

We may now prove our first oscillation result for (1.1).

Theorem 3.2. *Assume that* $p_1(t)$ *satisfies condition (A)* , (1.2) *holds,* $q \in \mathcal{R}^+$, *and*

$$\int_T^\infty \frac{1}{r(t)}\Delta t = \int_T^\infty p_1(s)\Delta s = \infty.$$

Then all solutions of (1.1) *are oscillatory.*

Proof. Let us suppose that x is a solution of (1.1) and to be specific, suppose that $x(t) > 0$ for large t, since the other case is similar. In view of the Lemma 2.3, we may then suppose also that $x^\Delta(t) > 0$, for $t \geq T \geq T_0$. Multiplying (3.1) by

$$\frac{1}{f(x(\sigma(t)))}$$

and integrating by parts (Lemma 2.2) for $t \geq T$ gives

$$\frac{r(t)x^\Delta(t)}{f(x(t))} - \int_T^t r(s)x^\Delta(s)\left(\frac{1}{f \circ x}\right)^\Delta(s)\Delta s + \int_T^t p_1(s)\Delta s = \frac{r(T)x^\Delta(T)}{f(x(T))} \tag{3.11}$$

for $t \geq T$. We note from the chain rule (Lemma 2.1) and quotient rule that

$$\begin{aligned} \int_T^t r(s)x^\Delta(s)\left(\frac{1}{f \circ x}\right)^\Delta(s)\Delta s &= -\int_T^t r(s)x^\Delta(s)\frac{(f \circ x)^\Delta(s)}{f(x(s))f(x(\sigma(s)))}\Delta s \\ &= -\int_T^t r(s)x^\Delta(s)\left\{\int_0^1 f'(x(s)+h\mu(s)x^\Delta(s))dh\right\}\frac{x^\Delta(s)}{f(x(s))f(x(\sigma(s)))}\Delta s \\ &\leq 0 \end{aligned} \tag{3.12}$$

since $f'(x) > 0$, $x \neq 0$. Consequently, from (3.11) and (3.12) we have

$$\frac{r(t)x^\Delta(t)}{(f \circ x)(t)} + \int_T^t p_1(s)\Delta s \leq \frac{r(T)x^\Delta(T)}{(f \circ x)(T)} \tag{3.13}$$

But now the left side of (3.13) is unbounded and the right side is bounded. This contradiction proves the theorem. □

Theorem 3.2 extends an old result of Fite [14] for the linear second order differential equation $x'' + p(t)x = 0$ which says that all solutions oscillate if

$$\int^\infty p(t)dt = \infty.$$

This result was subsequently extended by a number of authors to the nonlinear ordinary differential equation and to certain nonlinear dynamic equations on time scales. In the case $\mathbb{T} = \mathbb{R}$, Waltman [18] obtained the Fite result and in the time scales case, this result (the Leighton–Wintner result [16], [19]) may be found in [2] (for the case $p(t) > 0$) and in [10] and [11] with no explicit sign assumption on $p(t)$.

Our next oscillation result extends a result of Atkinson [1] for the nonlinear second order differential equation.

Theorem 3.3. *Assume that* $p_1(t)$ *satisfies condition (A),* (1.2) *holds,* $q(t) \geq 0$ *(so that* $q \in \mathcal{R}^+$*) and that*

$$\int_T^\infty \frac{1}{r(t)} \Delta t = \infty.$$

Suppose also that

$$\limsup_{t \to \infty} \frac{1}{r(t)} \int_T^t \sigma(s) p_1(s) \Delta s = +\infty,$$

and that the following nonlinearity condition holds:

$$\int_{\pm 1}^{\pm\infty} \frac{du}{f(u)} < \infty. \tag{3.14}$$

Then all solutions of (1.1) *are oscillatory.*

Proof. We may assume, without loss of generality, (as in the proof of Theorem 3.2), that $x(t)$ is a solution of (1.1) with $x(t) > 0$, and $x^\Delta(t) > 0$ for all $t \geq T > 0$. We define

$$H(t) := \frac{t}{f(x(t))},$$

then multiplying the first term of (3.1) by $H(\sigma(t))$ and integrating gives

$$\begin{aligned}
&\int_T^t H(\sigma(s))(r x^\Delta)^\Delta(s) \Delta s \\
&= H(t) r(t) x^\Delta(t) - H(T) r(T) x^\Delta(T) - \int_T^t r(s) x^\Delta(s) H^\Delta(s) \Delta s \\
&= H(t) r(t) x^\Delta(t) - H(T) r(T) x^\Delta(T) \\
&- \int_T^t r(s) x^\Delta(s) \left(\frac{f(x(s)) - s(f \circ x)^\Delta(s)}{f(x(s)) f(x(\sigma(s)))} \right) \Delta s \\
&= H(t) r(t) x^\Delta(t) - H(T) r(T) x^\Delta(T) \\
&- \int_T^t \frac{r(s) x^\Delta(s)}{f(x(\sigma(s)))} \Delta s + \int_T^t \frac{H(s) r(s) x^\Delta(s) (f \circ x)^\Delta(s)}{f(x(\sigma(s)))} \Delta s.
\end{aligned}$$

Now

$$\begin{aligned}
&\int_T^t \frac{H(s) r(s) x^\Delta(s) (f \circ x)^\Delta(s)}{f(x(\sigma(s)))} \Delta s \\
&= \int_T^t \frac{H(s) r(s) x^\Delta(s)}{f(x(\sigma(s)))} \left(\int_0^1 f'(x_h(s)) dh \right) x^\Delta(s) \Delta s \\
&> 0,
\end{aligned}$$

since $f'(x) > 0$, where

$$x_h(s) := x(s) + h\mu(s) x^\Delta(s) > 0.$$

Therefore, we get

$$\begin{aligned} &\int_T^t H(\sigma(s))(rx^\Delta)^\Delta(s)\Delta s \\ \geq\ & H(t)r(t)x^\Delta(t) - H(T)r(T)x^\Delta(T) - \int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s. \end{aligned} \tag{3.15}$$

From equation (3.1) we have after a multiplication by $H(\sigma(t))$ and an integration

$$\int_T^t H(\sigma(s))(rx^\Delta)^\Delta(s)\Delta s + \int_T^t \sigma(s)p_1(s)\Delta s = 0 \tag{3.16}$$

and so by (3.15) we have after rearranging

$$\begin{aligned} &H(t)r(t)x^\Delta(t) + \int_T^t \sigma(s)p_1(s)\Delta s \\ \leq\ & H(T)r(T)x^\Delta(T) + \int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s. \end{aligned} \tag{3.17}$$

Now since $r^\Delta(t) = q(t)e_q(t,t_0) \geq 0$, and $x^\Delta(t) > 0$, it follows that

$$\int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s \leq r(t)\int_T^t \frac{x^\Delta(s)}{f(x(\sigma(s)))}\Delta s. \tag{3.18}$$

We now define the function

$$G(u) := \int_{u_0}^u \frac{1}{f(s)}ds, \quad (u_0 := x(T) > 0), \tag{3.19}$$

so that $G'(u) = \frac{1}{f(u)}$. Then by the chain rule (2.1)

$$(G(x(t)))^\Delta = \left(\int_0^1 \frac{1}{f(x_h(t))}dh\right)x^\Delta(t) \geq \left(\int_0^1 \frac{1}{f(x(\sigma(t)))}dh\right)x^\Delta(t) \tag{3.20}$$

since $x_h(t) \leq x(\sigma(t))$, and therefore

$$\frac{1}{f(x_h(t))} \geq \frac{1}{f(x(\sigma(t)))}.$$

Consequently, we have

$$(G(x(t)))^\Delta \geq \frac{x^\Delta(t)}{f(x(\sigma(t)))}. \tag{3.21}$$

Now since $x^\Delta(t) > 0$, we have

$$\lim_{t\to\infty} x(t) := L \leq \infty.$$

Therefore,

$$\lim_{t\to\infty} G(x(t)) = \lim_{t\to\infty}\int_{u_0}^{x(t)} \frac{du}{f(u)} = \int_{u_0}^L \frac{du}{f(u)} := L_1 < \infty. \tag{3.22}$$

On the other hand, we also have

$$\int_T^t (G(x(s)))^\Delta \Delta s = G(x(t)) - G(x(T))$$

and so from (3.18), (3.21), and (3.22) we have

$$\int_T^t \frac{r(s)x^\Delta(s)}{f(x(\sigma(s)))}\Delta s \le r(t)L_2,$$

where $L_2 := -G(x(T)) + L_1$. Now using this in (3.17) we get

$$H(t)x^\Delta(t) + \frac{1}{r(t)}\int_T^t \sigma(s)p_1(s)\Delta s \le \frac{H(T)r(T)x^\Delta(T)}{r(t)} + L_2.$$

Hence

$$\begin{aligned} &\frac{tx^\Delta(t)}{f(x(t))} + \frac{1}{r(t)}\int_T^t \sigma(s)p_1(s)\Delta s - L_2 \\ \le\ &\frac{Tr(T)x^\Delta(T)}{r(t)f(x(T))} \le \frac{Tx^\Delta(T)}{f(x(T))}, \end{aligned} \tag{3.23}$$

(since $r(t) \ge r(T)$).
But now if we take the lim sup on both sides of (3.23), we see that the left side is not bounded above and this is a contradiction. This proves the Theorem. □

If $q(t) \equiv 0$, then equation (1.1) becomes

$$x^{\Delta\Delta} + p(t)(f \circ x^\sigma) = 0, \tag{3.24}$$

In this case we have the following

Corollary 3.4. *Suppose that* (1.2) *and* (3.14) *hold,*

$$\limsup_{t\to\infty} \int_T^t \sigma(s)p(s)\Delta s = +\infty$$

and that p satisfies condition (A). Then all solutions of (3.24) *are oscillatory.*

This Corollary extends the differential equations result of Atkinson [1] which says that

$$\int^\infty tp(t)dt = \infty$$

implies that all solutions of the equation

$$x'' + p(t)x^{2n+1} = 0$$

are oscillatory, where $p(t) > 0$ and is continuous on $[a, +\infty)$. Notice that no explicit sign assumptions on $p(t)$ are necessary in Corollary 3.4. The extension of the Atkinson result to time scales was also obtained in [10], [11], and, for the case $p(t) > 0$, in [2].

If we do not assume the nonlinearity condition (3.14) in the previous theorem, then we can conclude that all bounded solutions are oscillatory. That is, we have

Corollary 3.5. *Assume that $p_1(t)$ satisfies condition (A),* (1.2) *holds, $q(t) \geq 0$ and that*

$$\int_T^\infty \frac{1}{r(t)} \Delta t = \infty.$$

Suppose also that

$$\limsup_{t \to \infty} \frac{1}{r(t)} \int_T^t \sigma(s) p_1(s) \Delta s = +\infty.$$

Then all bounded solutions of (1.1) *are oscillatory.*

We next want to show how a generalized Riccati transformation may be used to establish some additional new oscillation criteria. This idea was also used in [5] and [2]. We shall first introduce the following condition:

We say that a function $g : \mathbb{T} \to \mathbb{R}$ satisfies **condition (B)** if for each $k > 0$ there exists $m > 0$ such that $g(x) \geq m$ provided $x \geq k$. This condition says that $g(x)$ is bounded away from 0 if x is.

The following Lemma may be found in [2].

Lemma 3.6. *If z and x are differentiable on $\mathbb{T}$ and $x(t) \neq 0$ on $\mathbb{T}$, then*

$$x^\Delta \left(\frac{z^2}{x} \right)^\Delta = \left(z^\Delta \right)^2 - x x^\sigma \left[\left(\frac{z}{x} \right)^\Delta \right]^2.$$

We shall also need the following result.

Lemma 3.7. *Suppose that x is a solution of* (1.1) *and $x(t)x^\Delta(t) > 0$ for all $t \geq T_0$, and assume z and $f \circ x$ are differentiable functions on $\mathbb{T}$ with $xf(x) \neq 0$, $x \neq 0$. If we define*

$$w := \frac{z^2 r x^\Delta}{f \circ x},$$

then w satisfies

$$-w^\Delta = p_1 (z^\sigma)^2 - \frac{r x^\Delta (z^\Delta)^2}{(f \circ x)^\Delta} + \frac{r x^\Delta (f \circ x)(f \circ x^\sigma)}{(f \circ x)^\Delta} \left[\left(\frac{z}{f \circ x} \right)^\Delta \right]^2.$$

Proof. Let x, z, and w be as in the statement of this theorem. Then we have

$$\begin{aligned}
-w^\Delta &= -\left(\frac{z^2 r x^\Delta}{f \circ x} \right)^\Delta \\
&= -\left(\frac{z^2}{f \circ x} \right)^\sigma (r x^\Delta)^\Delta - \left(\frac{z^2}{f \circ x} \right)^\Delta r x^\Delta \\
&= p_1 (z^2)^\sigma - \left(\frac{z^2}{f \circ x} \right)^\Delta r x^\Delta \\
&= p_1 (z^2)^\sigma - \frac{r x^\Delta}{(f \circ x)^\Delta} \left[(f \circ x)^\Delta \left(\frac{z^2}{f \circ x} \right)^\Delta \right].
\end{aligned}$$

Now using Lemma 3.6 with x replaced by $f \circ x$, we get

$$(f\circ x)^{\Delta}\left(\frac{z^2}{f\circ x}\right)^{\Delta} = (z^{\Delta})^2 - (f\circ x)(f\circ x)^{\sigma}\left[\left(\frac{z}{f\circ x}\right)^{\Delta}\right]^2.$$

Therefore, using this we have

$$-w^{\Delta} = p_1(z^2)^{\sigma} - \frac{rx^{\Delta}}{(f\circ x)^{\Delta}}(z^{\Delta})^2 + \frac{rx^{\Delta}(f\circ x)(f\circ x)^{\sigma}}{(f\circ x)^{\Delta}}\left[\left(\frac{z}{f\circ x}\right)^{\Delta}\right]^2$$

which proves the lemma. □

We can now state and prove the following result which replaces the nonlinear assumption (3.14) by the assumption that f' satisfies condition (B).

Theorem 3.8. *Assume that* (1.2) *holds, f' satisfies condition (B), and p_1 satisfies condition (A). Assume further that*

$$\int_T^{\infty}\frac{1}{r(t)}\Delta t = \infty$$

and that there exists a differentiable function z such that for all $K > 0$

$$\limsup_{t\to\infty}\int_a^t [p_1(z^{\sigma})^2 - Kr(z^{\Delta})^2]\Delta s = +\infty. \tag{3.25}$$

Then all solutions of (1.1) *oscillate.*

Proof. If not, then there is a solution x such that $x(t) > 0$ for $t \geq T_0$ for some sufficiently large T_0. Then by Lemma 3.1 there is a $T_1 \geq T_0$ such that $x^{\Delta}(t) > 0$ for $t \geq T_1$. As in Lemma 3.7, let

$$w := \frac{z^2 rx^{\Delta}}{f\circ x}$$

and note that $w(t) > 0$ for $t \geq T_1$ and from Lemma 3.7

$$\begin{aligned} -w^{\Delta} &= p_1(z^{\sigma})^2 - \frac{rx^{\Delta}(z^{\Delta})^2}{(f\circ x)^{\Delta}} + \frac{rx^{\Delta}(f\circ x)(f\circ x)^{\sigma}}{(f\circ x)^{\Delta}}\left[\left(\frac{z}{f\circ x}\right)^{\Delta}\right]^2 \\ &\geq p_1(z^{\sigma})^2 - \frac{rx^{\Delta}(z^{\Delta})^2}{(f\circ x)^{\Delta}}, \quad (\text{since } x^{\Delta} > 0), \end{aligned}$$

Therefore,

$$\begin{aligned} -\int_{T_1}^t w^{\Delta}(s)\Delta s &= -w(t) + w(T_1) \\ &\geq \int_{T_1}^t p_1(s)(z^{\sigma}(s))^2\Delta s - \int_{T_1}^t \frac{rx^{\Delta}(z^{\Delta}(s))^2}{(f\circ x)^{\Delta}(s)}\Delta s. \end{aligned} \tag{3.26}$$

Now from the Keller chain rule (Theorem 2.1)

$$(f\circ x)^{\Delta} = \left(\int_0^1 f'(x_h(t)dh\right)x^{\Delta}(t),$$

where

$$x_h(t) = x(t) + h\mu(t)x^\Delta(t) \geq x(t) \geq x(T_1)$$

since $x^\Delta(t) > 0$. Since f' satisfies condition (B) corresponding to $k := x(T_1) > 0$, there is an $m > 0$ such that

$$f'(x_h(t)) \geq m, \quad \text{for all} \quad t \geq T_1.$$

Therefore

$$\frac{(f \circ x)^\Delta}{x^\Delta} = \int_0^1 f'(x_h(t))dh \geq m \int_0^1 dh = m.$$

Hence

$$0 < \frac{x^\Delta}{(f \circ x)^\Delta} \leq \frac{1}{m}, \quad t \geq T_1.$$

It follows that

$$-\int_{T_1}^t \frac{r(s)x^\Delta(s)(z^\Delta(s))^2}{(f \circ x)^\Delta(s)}\Delta s \geq -\frac{1}{m}\int_{T_1}^t r(s)(z^\Delta(s))^2\Delta s,$$

and so we have from (3.26) that

$$\begin{aligned} w(T_1) &\geq -w(t) + w(T_1) = -\int_{T_1}^t w^\Delta(s)\Delta s \\ &\geq \int_{T_1}^t p_1(s)(z^\sigma(s))^2\Delta s - \frac{1}{m}\int_{T_1}^t r(s((z^\Delta(s))^2\Delta s \\ &= \int_{T_1}^t \left[p_1(s)(z^\sigma(s))^2 - Kr(s)(z^\Delta(s))^2\right]\Delta s \quad (K := \frac{1}{m}) \end{aligned}$$

which leads (using (3.25)) to a contradiction and the proof is complete. □

Note that if $f(x) = x$, then f' satisfies the condition (B) so Theorem 3.8 applies to the linear case as well, in contrast to Theorem 3.3.

4. Examples and Remarks

Remark 4.1. In the paper [10] the nonlinearity was assumed to satisfy (1.4) (in [11], (1.4) was assumed to hold for all large x.). In the paper [2] the case $f(x) = x^\gamma$, $\gamma > 1$, was considered which also satisfies (1.4) for all $x \neq 0$. It is not difficult to give an example of a function which satisfies the nonlinearity condition (3.14) with $xf(x) > 0$ and $f'(x) > 0$ for $x \neq 0$, but which does not satisfy (1.4). To see this, define f for $x \geq 0$ by $f(0) = 0$ and for $x > 0$, and $n = 0, 1, 2, \cdots$,

$$f(x) = \begin{cases} x^2 + f(2n), & 2n \leq x \leq 2n+1 \\ f(2n+1) + \frac{1}{2n+2}(x - (2n+1)), & 2n+1 \leq x \leq 2n+2. \end{cases}$$

We then define $f(-x) = -f(x)$ for $x > 0$. Then it is easy to show that

$$f(x) \geq \frac{x^2}{2}, \quad \text{for} \quad x \geq 3,$$

and we have

$$f'(x) = \frac{1}{2n+2} \quad \text{for} \quad 2n+1 < x < 2n+2.$$

Hence, $\liminf_{x\to\infty} f'(x) = 0$ so condition (B) does not hold, but (3.14) does. Hence, $\int_{\pm 1}^{\pm\infty} \frac{du}{f(u)} < \infty$, i.e., (3.14) holds, but (1.4) does not hold.

Example 4.2. We let $q(t) \equiv 0$ and

$$p(t) := \frac{\lambda}{t^\alpha \sigma(t)} + \frac{\beta(-1)^t}{t^\alpha}, \quad t \in \mathbb{T} = \mathbb{N},$$

where $\beta, \lambda > 0$, $0 < \alpha < 1$. We note that

$$\int_t^\infty \frac{\Delta s}{s^\alpha \sigma(s)} = \sum_{k=n}^\infty \frac{1}{k^\alpha (k+1)} \sim \frac{1}{\alpha n^\alpha}, \quad \text{large } n$$

in the sense that for any $0 < \varepsilon < 1$ there exists $N \geq 1$ such that

$$\frac{1-\varepsilon}{\alpha} \leq n^\alpha \sum_{k=n}^\infty \frac{1}{k^\alpha (k+1)} \leq \frac{1+\varepsilon}{\alpha}$$

for all $n \geq N$. Also, we have

$$\left| \int_t^\infty \frac{(-1)^s}{s^\alpha} \Delta s \right| = \left| \sum_{k=n}^\infty \frac{(-1)^k}{k^\alpha} \right| \leq \frac{1}{n^\alpha}.$$

Therefore, if $\frac{\lambda}{\alpha} > \beta$, then $p(t)$ satisfies condition (A). Also, as above, one can show that

$$-\infty = \liminf_{t\to\infty} \int_a^t \sigma(s) p(s) \Delta s < \limsup_{t\to\infty} \int_a^t \sigma(s) p(s) \Delta s = \infty \tag{4.1}$$

provided $\lambda < \beta(1-\alpha)$. Hence, if

$$\beta\alpha < \lambda < \beta(1-\alpha),$$

and if (1.2) and (3.14) hold, then all solutions of

$$x^{\Delta\Delta} + p(t)(f \circ x^\sigma) = 0$$

are oscillatory on $\mathbb{N}$ by Corollary 3.4. Note that this includes the superlinear case, but does not treat the linear and sublinear case. It turns out that one can also apply Theorem 3.8 to this example as well if f' satisfies condition (B), say instead of (3.14). However none of the references can be applied since in all of these $p(t)$ is assumed to be positive.

Example 4.3. Let $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ and consider the q-difference equation

$$x^{\Delta\Delta} + p(t)(f \circ x^\sigma) = 0, \tag{4.2}$$

where f satisfies the nonlinearity condition (3.14). Assume that $1 < \alpha < 2$, let $m := \frac{q^{\alpha-1}-1}{q^{\alpha+1}+1}$, and assume further that $\beta > 0$ and $0 < m\alpha < \lambda$. Define

$$p(t) := \frac{1}{t^\alpha} \left(\lambda + \beta(-1)^n \right), \quad t = q^n \in \mathbb{T}.$$

Then we have

$$\begin{aligned}\int_{t_n}^{\infty} p(s)\Delta s &= \sum_{k=n}^{\infty} \int_{t_k}^{\sigma(t_k)} p(s)\Delta s \\ &= \sum_{k=n}^{\infty} p(t_k)\mu(t_k) = \sum_{k=n}^{\infty} \frac{(\lambda+\beta(-1)^k)}{q^{\alpha k}}(q-1)q^k \\ &= (q-1)\sum_{k=n}^{\infty}(\lambda+\beta(-1)^k)(q^{1-\alpha})^k \\ &= \frac{(q-1)q^{\alpha-1}}{q^{n(\alpha-1)}}\left(\frac{\lambda}{q^{\alpha-1}-1}+\frac{\beta(-1)^n}{q^{\alpha+1}-1}\right).\end{aligned}$$

Notice that this last expression is nonnegative since

$$\frac{\lambda}{\beta} \geq \frac{q^{\alpha-1}-1}{q^{\alpha+1}+1} = m.$$

Hence, $p(t)$ satisfies condition (A). Consider next

$$\begin{aligned}\int_{1}^{t_n} \sigma(s)p(s)\Delta s &= \sum_{k=1}^{n} \frac{q^{k+1}(\lambda+\beta(-1)^k)}{q^{\alpha k}}(q-1)q^k \\ &= q(q-1)\sum_{k=1}^{n} q^{(2-\alpha)k}(\lambda+\beta(-1)^k).\end{aligned}$$

It follows that $\limsup_{n\to\infty}\int_1^{t_n}\sigma(s)p(s)\Delta s = +\infty$, and so all solutions are oscillatory on $\mathbb{T}=q^{\mathbb{N}_0}$ by Corollary 3.4. Note also that if $0<\lambda<\beta$, we have

$$-\infty = \liminf_{n\to\infty}\int_1^{t_n}\sigma(s)p(s)\Delta s < \limsup_{n\to\infty}\int_1^{t_n}\sigma(s)p(s)\Delta s = \infty.$$

Again we notice that $p(t)$ is not eventually of one sign in this example so the criteria in the references do not apply.

Example 4.4. As a final example for the case when $\mathbb{T}$ is the real interval $[1,\infty)$, suppose f satisfies (1.2), and the nonlinearity condition (3.14). Let

$$p(t) := \frac{\lambda}{t^{1+\alpha}} + \frac{\beta \sin t}{t^{\alpha}},$$

where λ, α, β are all positive numbers and satisfy

$$\beta\alpha < \lambda, \quad 0<\alpha<1. \tag{4.3}$$

Then one can show that $\int_t^\infty p(s)ds \geq 0$ for all large t. Moreover, we also have

$$\limsup_{t\to\infty}\int_a^t sp(s)ds = +\infty,$$

since $\lambda>0$. (If, in addition, $\lambda<\beta(1-\alpha)$, then we also have

$$\liminf_{t\to\infty}\int_a^t sp(s)ds = -\infty.)$$

Therefore if (4.3) holds, then all solutions of the nonlinear differential equation $x'' + p(t)f(x) = 0$ are oscillatory on $[1, \infty)$ by Corollary 3.4. This may also be deduced from Theorem 3.8 by choosing $z(t) = \sqrt{t}$.

If

$$p(t) := \frac{\lambda}{t^{\alpha_1}} + \frac{\beta \sin t}{t^{\alpha_2}},$$

then as above $p(t)$ satisfies condition (A) if λ, β, α_1, α_2 are positive numbers satisfying $0 < \alpha_1 \le \alpha_2 < 1$ with $\frac{\lambda}{\alpha_1} > \beta$ if $\alpha_1 = \alpha_2$. Moreover, $\limsup_{t\to\infty} \int_1^t sp(s)ds = +\infty$, since $\alpha_1 \le \alpha_2 < 1$ and so oscillation of $x'' + p(t)f(x)$ is a consequence of Corollary 3.4 (or Theorem 3.8), where we again assume f satisfies (1.2), (3.14).

References

[1] F.V. Atkinson, On second-order nonlinear oscillations, Pacific J. Math., 5 (1955), 643–647.

[2] E. Akın-Bohner, M. Bohner, and S.H. Saker, Oscillation criteria for a certain class of second order Emden-Fowler dynamic equations, preprint.

[3] M. Bohner and A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[4] M. Bohner and S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, Rocky Mountain Journal of Mathematics, to appear.

[5] M. Bohner and S. H. Saker, Oscillation criteria for perturbed noninear dynamic equations, Mathematical and Computer Modeling, to appear.

[6] L. Erbe, Oscillation theorems for second order nonlinear differential equations, Proc. Amer. Math. Soc., 24 (1970), 811–814.

[7] L. Erbe, Oscillation criteria for second order linear equations on a time scale, Canadian Applied Mathematics Quarterly, 9 (2001), 1–31.

[8] L. Erbe, Oscillation criteria for second order nonlinear differential equations, Annali di Matematica pura ed applicata, XCIV (1972), 257–268.

[9] L. Erbe and A. Peterson. Oscillation criteria for second order matrix dynamic equations on a time scale, Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, J. Comput. Appl. Math., 141(1-2), (2002), 169–185.

[10] L. Erbe and A. Peterson, Boundedness and oscillation for nonlinear dynamic equations on a time scale, Proc. Amer. Math. Soc., 132 (2003), 735–744.

[11] L. Erbe and A. Peterson, An oscillation result for a nonlinear dynamic equation on a time scale, Canadian Applied Math Quarterly, to appear.

[12] L. Erbe, A. Peterson, and P. Rehak, Comparison Theorems for Linear Dynamic Equations on Time Scales, Journal of Mathematical Analysis and Applications, 275 (2002), 418–438.

[13] L. Erbe, A. Peterson, and S. H. Saker, Oscillation Criteria for second–order nonlinear dynamic equations on time scales, Journal of the London Mathematical Society, 67 (2003), 701–714.

[14] W.B. Fite, Concerning the zeros of solutions of certain differential equations, Trans. Amer. Math. Soc., 19 (1917), 341-352.

[15] S. Keller, Asymptotisches Verhalten Invarianter Faserbündel bei Diskretisierung und Mittelwertbildung im Rahmen der Analysis auf Zeitskalen, PhD thesis, Universität Augsburg, 1999.

[16] W. Leighton, On self-adjoint differential equations of second order, J. London Math. Soc., 27 (1952), 37–47.

[17] C. Pötzsche, Chain rule and invariance principle on measure chains, Special Issue on "Dynamic Equations on Time Scales", edited by R. P. Agarwal, M. Bohner, and D. O'Regan, J. Comput. Appl. Math., 141(1-2) (2002), 249-254.

[18] P. Waltman, An oscillation criterion for a nonlinear second order equation, J. Math. Anal. Appl., 10 (1965), 439–441.

[19] A. Wintner, On the nonexistence of conjugate points, Amer. J. Math., 73 (1951), 368–380.

Department of Mathematics, University of Missouri–Rolla

Department of Mathematics, University of Nebraska–Lincoln

Department of Mathematics, University of Nebraska–Lincoln

E-mail address: bohner@umr.edu

E-mail address: lerbe@math.unl.edu

E-mail address: apeterso@math.unl.edu