# Henstock–Kurzweil delta and nabla integrals

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**Abstract**

We will study the Henstock–Kurzweil delta and nabla integrals, which generalize the Henstock–Kurzweil integral. Many properties of these integrals will be obtained. These results will enable time scale researchers to study more general dynamic equations. The Hensock–Kurzweil delta (nabla) integral contains the Riemann delta (nabla) and Lebesque delta (nabla) integrals as special cases.

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## 1 Introduction

In this paper we shall study the so-called Henstock–Kurzweil delta and nabla integrals, which generalize the so-called delta and nabla integrals respectively, which have been widely used in the study of dynamic equations on a time scale. This will inable one to solve more general dynamic equations and hence will be of great interest to researchers in this area.

First we introduce the following concepts related to the notion of time scales. A time scale $\mathbb{T}$ is just any closed nonempty subset of the real numbers $\mathbb{R}$.

**Definition 1.1** *Let* $\mathbb{T}$ *be a time scale and define the* forward jump operator $\sigma(t)$ *at* $t$ *, for* $t \in \mathbb{T}$, *by*

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

*and the* backward jump *operator* $\rho(t)$ *at* $t$, *for* $t \in \mathbb{T}$, *by*

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}$$

*where* $\inf \emptyset := \sup \mathbb{T}$ *and* $\sup \emptyset := \inf \mathbb{T}$, *where* $\emptyset$ *denotes the empty set.*

We assume thoughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ we say t is right-dense, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$ we say t is left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous provided $f$ is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. We shall also use the notation $\mu(t) := \sigma(t) - t$, where $\mu$ is called the graininess function and $\nu(t) := t - \rho(t)$, where $\nu$ is called the left-graininess function. We denote the natural numbers by $\mathbb{N}$ and the nonnegative integers by $\mathbb{N}_0$.

**Definition 1.2** *Thoughout this paper we make the blanket assumption that $a, b \in \mathbb{T}$ and we define the* time scale interval *in $\mathbb{T}$ by*

$$[a, b]_{\mathbb{T}} := \{t \in \mathbb{T} \ \ such \ \ that \ \ a \leq t \leq b\}.$$

The notion of a time scale was introducted by S. Hilger [5]. Related work on the calculus of time scales may be found in Agarwal and Bohner [1], and Erbe and Hilger [4]. See also the introductory books on time scales [2], [3], and [6].

**Definition 1.3** *Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$, then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U$ of $t$ such that*

$$|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,$$

*for all $s \in U$. We call $x^{\Delta}(t)$ the* delta derivative *of $x$ at $t$.*

It can be shown that [2, Theorem 1.16] if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T}$=$\mathbb{N}_0$, then

$$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t),$$

here $\Delta$ is the forward difference operator [7]. If $t$ is right-dense, then it can be shown that [2, Theorem 1.16]

$$x^{\Delta}(t) = \lim_{s \to t} \frac{x(t) - x(s)}{t - s}.$$

In particular, if $\mathbb{T}$ is the real interval $[a, \infty)$, then $x^{\Delta}(t) = x'(t)$.

**Definition 1.4** *We say $\delta = (\delta_L, \delta_R)$ is a $\Delta$-gauge for $[a, b]_{\mathbb{T}}$ provided $\delta_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\delta_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\delta_L(a) \geq 0$, $\delta_R(b) \geq 0$, and $\delta_R(t) \geq \mu(t)$ for all $t \in [a, b)_{\mathbb{T}}$. Similarly we say $\gamma = (\gamma_L, \gamma_R)$ is a $\nabla$-gauge for $[a, b]_{\mathbb{T}}$ provided $\gamma_L(t) > 0$ on $(a, b]_{\mathbb{T}}$, $\gamma_R(t) > 0$ on $[a, b)_{\mathbb{T}}$, $\gamma_L(a) \geq 0$, $\gamma_R(b) \geq 0$, and $\gamma_L(t) \geq \nu(t)$ for all $t \in (a, b]_{\mathbb{T}}$.*

Since for a $\Delta$-gauge, $\delta$, we always assume $\delta_L(a) \geq 0$ and $\delta_R(b) \geq 0$, we will sometimes not even point this out. Similarly for a $\nabla$-gauge, $\gamma$, we will not always make the point that $\gamma_L(a) \geq 0$ and $\gamma_R(b) \geq 0$.

**Definition 1.5** *A partition $\mathcal{P}$ for $[a,b]_{\mathbb{T}}$ is a division of $[a,b]_{\mathbb{T}}$ denoted by*

$$\mathcal{P} = \{a = t_0 \leq \xi_1 \leq t_1 \leq \cdots \leq t_{n-1} \leq \xi_n \leq t_n = b\}$$

*with $t_i > t_{i-1}$ for $1 \leq i \leq n$ and $t_i, \xi_i \in \mathbb{T}$. We call the points $\xi_i$ tag points and the points $t_i$ end points. As in Peng–Yee [8], we sometimes denote such a partition by $\mathcal{P} = \{[u,v];\xi\}$, where $[u,v]$ denotes a typical interval in $\mathcal{P}$ and $\xi$ is the associated tag point in $[u,v]$.*

**Definition 1.6** *If $\delta$ is a $\Delta$-gauge for $[a,b]_{\mathbb{T}}$, then we say a partition $\mathcal{P}$ is $\delta$-fine if*

$$\xi_i - \delta_L(\xi_i) \leq t_{i-1} < t_i \leq \xi_i + \delta_R(\xi_i)$$

*for $1 \leq i \leq n$. Similarly, if $\gamma$ is a $\nabla$-gauge for $[a,b]_{\mathbb{T}}$, then we say the partition $\mathcal{P}$ is $\gamma$-fine if*

$$\xi_i - \gamma_L(\xi_i) \leq t_{i-1} < t_i \leq \xi_i + \gamma_R(\xi_i)$$

*for $1 \leq i \leq n$.*

Now we can define the Henstock–Kurzweil delta and nabla integral.

**Definition 1.7** *We say that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock–Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$ with value $I = HK\int_a^b f(t)\Delta t$, provided given any $\epsilon > 0$ there exists a $\Delta$-gauge, $\delta$, for $[a,b]_{\mathbb{T}}$ such that*

$$|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})| < \epsilon$$

*for all $\delta$-fine partitions $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$. Similarly we say that $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is Henstock–Kurzweil nabla integrable on $[a,b]_{\mathbb{T}}$ with value $I = HK\int_a^b f(t)\nabla t$, provided given any $\epsilon > 0$ there exists a $\nabla$-gauge, $\gamma$, for $[a,b]_{\mathbb{T}}$ such that*

$$|I - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})| < \epsilon$$

*for all $\gamma$-fine partitions $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$.*

**Remark 1.8** *If $f$ is Riemann delta integrable on $[a,b]_{\mathbb{T}}$ according to the Riemann sums definition given for a bounded function $f(t)$ on $[a,b]_{\mathbb{T}}$ in Definition 5.10 in [3], then it is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$HK\int_a^b f(t)\Delta t = \int_a^b f(t)\Delta t.$$

The proof of this remark follows from the fact that if we define $\delta(t) = (\delta_L(t), \delta_R(t))$, by $\delta_L(t) = \delta_L > 0$ a constant for $t \in [a,b]_{\mathbb{T}}$ and $\delta_R(t) = \delta_L$ for all right-dense points in $[a,b]_{\mathbb{T}}$ and $\delta_R(t) = \mu(t)$ for all right-scattered points in $[a,b]_{\mathbb{T}}$, then $\delta$ is a $\Delta$-gauge of $[a,b]_{\mathbb{T}}$.

As in Lee Peng–Yee [8] we sometimes in proofs use the abbreviation

$$\sum f(\xi)(u-v) := \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1}).$$

For the Henstock–Kurzweil delta and nabla integrals to make any sense we prove the following lemma. In this lemma and later we use the notation, if $s \in [a,b]$, then

$$\beta_L(s) := \sup\{t \in [a,b]_{\mathbb{T}} : t \le s\}, \quad \beta_R(s) := \inf\{t \in [a,b]_{\mathbb{T}} : t \ge s\}.$$

**Lemma 1.9** *If $\delta$ is a $\Delta$-gauge for $[a,b]_{\mathbb{T}}$, then there is a $\delta$-fine partition $\mathcal{P}$ for $[a,b]_{\mathbb{T}}$. Similarly if $\gamma$ is a $\nabla$-gauge for $[a,b]_{\mathbb{T}}$, then there is a $\gamma$-fine partition $\mathcal{P}$ for $[a,b]_{\mathbb{T}}$.*

*Proof:* We just prove the first statement. Assume there is no such $\delta$-fine partition $\mathcal{P}$ for $[a,b]_{\mathbb{T}}$. Let $c = \frac{1}{2}(b-a)$ and let

$$d = \beta_L(c), \quad e = \beta_R(c).$$

Then either $[a,d]_{\mathbb{T}}$ or $[e,b]_{\mathbb{T}}$ does not have a $\delta$-fine partition. Assume $[a_1,b_1]_{\mathbb{T}}$ is one of these two intervals such that $[a_1,b_1]_{\mathbb{T}}$ does not have a $\delta$-fine partition. Repeating this argument we get a nested sequence of intervals $[a_i,b_i]_{\mathbb{T}}$ with $|b_i - a_i| \le \frac{b-a}{2^i}$, each of which does not have a $\delta$-fine partition. Let

$$\xi_0 = \lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i.$$

Then for $i$ sufficiently large

$$\xi_0 - \delta_L(\xi_0) \le a_i < b_i \le \xi_0 + \delta_R(\xi_0).$$

But then

$$\{a_i \le \xi_0 \le b_i\}$$

is a $\delta$-fine partition of $[a_i,b_i]_{\mathbb{T}}$, which is a contradiction. $\square$

We next give an interesting introductory example. In this example we use the fact [2, Theorem 1.79] that if $[c,d]_{\mathbb{T}} = \{t_0 = c, t_1, t_2, \cdots, t_n = d\}$, then

$$\int_c^d f(t)\Delta t = \sum_{i=0}^{n-1} f(t_i)(t_{i+1} - t_i) = \sum_{i=0}^{n-1} f(t_i)\mu(t_i).$$

**Example 1.10** *Let $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Let $f : \mathbb{T} \to \mathbb{R}$ be defined by*

$$f(t) = \begin{cases} \frac{1}{\sqrt{t}+\sqrt{\sigma(t)}}, & t \ne 0 \\ 0, & t = 0, \end{cases}$$

*then we claim that $f$ is Henstock–Kurzweil delta integrable with*

$$HK\int_0^1 f(t)\Delta t = 1$$

*even though the delta integral $\int_0^1 f(t)\Delta t$ does not exist (although it does exist as an improper delta integral). To see that $f$ is Henstock–Kurzweil delta integrable on $[0,1]_{\mathbb{T}}$, let $0<\varepsilon<1$ be given. Assume that $\delta$ is a $\Delta$-gauge on $[0,1]_{\mathbb{T}}$ satisfying $\delta_L(t)=\frac{1}{2}\nu(t)$ for $t\in(0,1]_{\mathbb{T}}$ and $\delta_R(t)=\mu(t)$, for $t\in(0,1)_{\mathbb{T}}$, with $\delta_R(0)=\varepsilon^2$. Let $\mathcal{P}$ be a $\delta$-fine partition of $[0,1]_{\mathbb{T}}$. Since $\delta_L(t)=\frac{1}{2}\nu(t)$ on $(0,1]_{\mathbb{T}}$, we have that our first tag point is $\xi_1=0$. Also for $1\le i\le n-1$ we have $\delta_R(t_i)=\mu(t_i)$ and $\delta_L(\sigma(t_i))=\frac{1}{2}\nu(\sigma(t_i))$ which implies that*

$$\xi_{i+1}=t_i, \quad \textit{and} \quad t_{i+1}=\sigma(t_i), \quad 1\le i\le n-1.$$

*Now consider*

$$\begin{aligned}
\left|1-\sum_{i=1}^n f(\xi_i)[t_i-t_{i-1}]\right| &= \left|1-\sum_{i=2}^n f(\xi_i)[t_i-t_{i-1}]\right| \\
&= \left|1-\sum_{i=2}^n f(t_{i-1})\mu(t_{i-1})]\right| \\
&= \left|1-\int_{t_1}^1 f(t)\Delta t\right| \\
&= \left|1-[\sqrt{t}]_{t_1}^1\right| \\
&= \sqrt{t_1}\le\sqrt{\delta_R(0)}=\varepsilon,
\end{aligned}$$

*where we used the fact that $F(t)=\sqrt{t}$ is a delta antiderivative of $f(t)$ on $(0,1)_{\mathbb{T}}$.*

**Example 1.11** *Assume $[a,b)_{\mathbb{T}}$ contains a countable infinite subset $\cup_{i=1}^{\infty}\{r_i\}$ with $\sigma(r_i)=r_i$. Define $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$ by*

$$f(t)=\begin{cases}1, & t=r_i\\ 0, & t\neq r_i.\end{cases}$$

*Let $\varepsilon>0$ be given. Then define a $\Delta$-gauge, $\delta$, on $[a,b]_{\mathbb{T}}$ by $\delta_L(r_i)=\delta_R(r_i)=\frac{\varepsilon}{2^{i+2}}$, $i\ge 1$, $\delta_R(t)=\max\{1,\mu(t)\}$ and $\delta_L(t)=1$ for $t\in[a,b]_{\mathbb{T}}\setminus\cup_{i=1}^{\infty}\{r_i\}$. Let $\mathcal{P}$ be a $\delta$-fine partition of $[a,b]_{\mathbb{T}}$, then*

$$\begin{aligned}
|\sum_{i=1}^n f(\xi_i)(t_i-t_{i-1})| &\le \sum_{i=1}^{\infty}(\delta_L(r_i)+\delta_R(r_i)) \\
&= \sum_{i=1}^{\infty}\frac{\varepsilon}{2^{i+1}}<\varepsilon.
\end{aligned}$$

*Hence, even though in many cases $f$ is not delta integrable on $[a,b]_{\mathbb{T}}$, we get $f$ is Henstock–Kurzweil delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$HK\int_a^b f(t)\Delta t = 0.$$

*Similarly, one can show that if above $\cup_{i=1}^{\infty}\{r_i\}$ is a countable infinite subset of left dense points in $(a,b]_{\mathbb{T}}$, then $f$ as given above is Henstock–Kurtzweil nabla integrable on $[a,b]_{\mathbb{T}}$ with*

$$HK\int_a^b f(t)\nabla t = 0.$$

# 2 Main Results

The results in the following theorem in the special case of delta and nabla integrals are used all the time in the study of dynamic equations on time scales.

**Theorem 2.1** *Assume $f:[a,b]_{\mathbb{T}}\to\mathbb{R}$. If $f$ is HK-delta integrable on $[a,b]_{\mathbb{T}}$, then the value of the integral $HK\int_a^b f(t)\Delta t$ does not depend on $f(b)$. On the other hand if $c\in[a,b)_{\mathbb{T}}$ and $c$ is right–scattered, then $HK\int_a^b f(t)\Delta t$ does depend on the value $f(c)\mu(c)$. Also, if $f$ is HK-nabla integrable on $[a,b]_T$, then the value of the integral $HK\int_a^b f(t)\nabla t$ does not depend on $f(a)$ and if $c\in(a,b]_{\mathbb{T}}$ and $c$ is left–scattered, then $HK\int_a^b f(t)\Delta t$ does depend on the value $f(c)\nu(c)$.*

*Proof:* We will just prove the statements concerning the $HK$-delta integrals. Assume that $f$ is HK-delta integrable on $[a,b]_{\mathbb{T}}$. We consider the two cases $\rho(b)<b$ and $\rho(b)=b$. If $\rho(b)<b$, we choose $\delta_L(b)<\nu(b)$. In this case $b$ is not a tag point for any $\delta$-fine partition and hence $HK\int_a^b f(t)\Delta t$ does not depend on the value $f(b)$. Next consider the case $\rho(b)=b$. In this case given any $\varepsilon>0$ we can choose

$$\delta_L(b)<\frac{\epsilon}{|f(b)|+1}$$

so if $b=\xi_n$ is a tag point, then

$$|f(\xi_n)(t_n-t_{n-1})| = |f(b)(b-t_{n-1})| \le |f(b)|\delta_L(b) < |f(b)|\frac{\varepsilon}{|f(b)|+1} < \varepsilon$$

and the result follows. Next assume that $c\in[a,b)_{\mathbb{T}}$ and c is right–scattered. From Theorem 2.12 we get that

$$\begin{aligned} & HK\int_a^b f(t)\Delta t \\ = \;& HK\int_a^c f(t)\Delta t + HK\int_c^{\sigma(c)} f(t)\Delta t + HK\int_{\sigma(c)}^b f(t)\Delta t \\ = \;& HK\int_a^c f(t)\Delta t + f(c)\mu(c) + HK\int_{\sigma(c)}^b f(t)\Delta t. \end{aligned}$$

Since the first and last terms do not depend on $f(c)\mu(c)$ we get the desired result. □

**Remark 2.2** *From the proof of the first statement in Theorem 2.1 we see that in the definition of the Henstock–Kurzweil delta integral we can without loss of generality assume that the last tag point satisfies $\xi_n\ne b$.*

In the proof of the next theorem we use the notation

$$N_\mu := \{z_j \in [a,b)_{\mathbb{T}} : \mu(z_j) > 0\}$$

and note that $N_\mu$ is a countable set.

**Theorem 2.3** *Assume $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is continuous, $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$, and there is a set $D$ with $N_\mu \subset D \subset [a,b]_{\mathbb{T}}^\kappa$ such that $F^\Delta(t) = f(t)$ for $t \in D$ and $[a,b]_{\mathbb{T}} \backslash D$ is countable, then $f$ is $HK$–delta integrable on $[a,b]_{\mathbb{T}}$ with*

$$HK \int_a^b f(t)\Delta t = F(b) - F(a).$$

*Proof:* By hypothesis there is a set $D$ with $N_\mu \subset D \subset [a,b]_{\mathbb{T}}^\kappa$ such that $F^\Delta(t) = f(t)$ for $t \in D$ and $[a,b]_{\mathbb{T}} \backslash D$ is countable. Let

$$Y := [a,b]_{\mathbb{T}} \backslash D = \{y_1, y_2, \cdots\},$$

which is countable (could be finite). Let $\varepsilon > 0$ be given. We now define a $\Delta$-gauge, $\delta$, for $[a,b]_{\mathbb{T}}$.

First let $t = z_i \in N_\mu$, then we define $\delta_R(z_i) = \mu(z_i)$. Since $F$ is delta differentiable at $z_i$, there is a $\delta_L^1(z_i) > 0$ such that

$$|F(\sigma(z_i)) - F(s) - F^\Delta(z_i)[\sigma(z_i) - s]| \le \frac{\varepsilon}{4(b-a)}[\sigma(z_i) - s] \tag{2.1}$$

for all $s \in [z_i - \delta_L^1(z_i), z_i]_{\mathbb{T}}$. Also, since $F$ is continuous at $z_i$, we get there is a $\delta_L^2(z_i) > 0$ such that

$$|F(z_i) - F(s) - F^\Delta(z_i)[z_i - s]| \le \frac{\varepsilon}{2^{i+2}} \tag{2.2}$$

for all $s \in [z_i - \delta_L^2(z_i), z_i]_{\mathbb{T}}$. Then we define

$$\delta_L(z_i) = \min\{\delta_L^1(z_i),\ \delta_L^2(z_i)\}$$

so that (2.1) and (2.2) both hold for $s \in [z_i - \delta_L(z_i), z_i]_{\mathbb{T}}$.

Next assume $t \in Y$, then $t = y_j$ for some $j$. In this case, since $F$ is continuous at $y_j$, there is an $\eta(y_j) > 0$ such that

$$|F(r) - F(s) - f(y_j)(r - s)| \le \frac{\varepsilon}{2^{j+2}} \tag{2.3}$$

for all $r, s \in [y_j - \eta(y_j), y_j + \eta(y_j)]_{\mathbb{T}}$. In this case we define

$$\delta_R(y_j) = \delta_L(y_j) = \eta(y_j).$$

Finally, consider the case $t \in D \backslash \mathbb{N}_\mu$. Since $F$ is differentiable at $t$ we get that there is an $\alpha(t) > 0$ such that

$$|F(t) - F(s) - F^\Delta(t)(t - s)| \le \frac{\varepsilon}{4(b-a)}|t - s| \tag{2.4}$$

for $s \in [t-\alpha(t), t+\alpha(t)]_{\mathbb{T}}$. In this case we define

$$\delta_L(t) = \delta_R(t) = \alpha(t).$$

Hence $\delta$ is a $\Delta$-gauge on $[a,b]_{\mathbb{T}}$.

Now assume $\mathcal{P}$ is a $\delta$-fine partition of $[a,b]_{\mathbb{T}}$. Consider

$$\begin{aligned} & \left| F(b) - F(a) - \sum_{i=1}^{n} f(\xi_i)[t_i - t_{i-1}] \right| \\ = & \left| \sum_{i=1}^{n} \{[F(t_i) - F(t_{i-1})] + f(\xi_i)[t_i - t_{i-1}]\} \right| \\ \leq & \sum_{i=1}^{n} \left| [F(t_i) - F(t_{i-1})] + f(\xi_i)[t_i - t_{i-1}] \right|. \end{aligned} \tag{2.5}$$

We now look at the terms in this last sum for the cases

$$\xi_i \in \mathbb{N}_\mu, \ \xi_i \in Y, \text{ and } \xi_i \in D \backslash N_\mu$$

respectively. First assume $\xi_i \in \mathbb{N}_\mu$, then $\xi_i = z_j$ for some $j$ and $\sigma(z_j) > z_j$. Then, since $\delta_R(\xi_i) = \delta_R(z_j) = \mu(z_j)$, we have that either $t_i = z_j$ or $t_i = \sigma(z_j)$. If $t_i = z_j$, then using (2.2) we have that

$$\begin{aligned} |F(t_i) - F(t_{i-1}) - f(\xi_i)[t_i - t_{i-1}]| &= |F(z_j) - F(t_{i-1}) - F^{\Delta}(z_j)[z_j - t_{i-1}]| \\ &\leq \frac{\varepsilon}{2^{j+4}}, \end{aligned} \tag{2.6}$$

where $s = t_{i-1} \in [z_j - \delta_L(z_j), z_j]_{\mathbb{T}}$. On the other hand if $t_i = \sigma(z_j)$, then using (2.1) with $s = t_{i-1}$ we get that

$$\begin{aligned} |F(t_i) - F(t_{i-1}) - f(\xi_i)[t_i - t_{i-1}]| &= |F(\sigma(z_j)) - F(t_{i-1}) - F^{\Delta}(z_j)[\sigma(z_j) - t_{i-1}] \\ &\leq \frac{\varepsilon}{4(b-a)}[\sigma(z_j) - t_{i-1}] \leq \frac{\varepsilon}{4(b-a)}[t_i - t_{i-1}]. \end{aligned} \tag{2.7}$$

Next assume $\xi_i \in Y$, so $\xi_i = y_j$ for some $j$. Then by (2.3) with $r = t_i$ and $s = t_{i-1}$

$$\begin{aligned} |F(t_i) - F(t_{i-1}) - f(\xi_i)[t_i - t_{i-1}]| &= |F(t_i) - F(t_{i-1}) - f(y_j)[t_i - t_{i-1}]| \\ &\leq \frac{\varepsilon}{2^{j+2}}. \end{aligned} \tag{2.8}$$

Finally, assume that $\xi_i \in D \backslash N_\mu$. Then by the triangle inequality and (2.4)

$$\begin{aligned} & |F(t_i) - F(t_{i-1}) - f(\xi_i)[t_i - t_{i-1}]| = |F(t_i) - F(t_{i-1}) - F^{\Delta}(\xi_i)[t_i - t_{i-1}]| \\ \leq & |F(t_i) - F(\xi_i) - F^{\Delta}(\xi_i)[t_i - \xi_i]| + |F(\xi_i) - F(t_{i-1}) - F^{\Delta}(\xi_i)[\xi_i - t_{i-1}]| \\ \leq & \frac{\varepsilon}{4(b-a)}[t_i - t_{i-1}]. \end{aligned} \tag{2.9}$$

The result now follows from (2.5)–(2.9). $\square$

In most papers concerning dynamic equations on time scales the author(s) define the Cauchy–Newton delta integral as follows: If $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is a delta antiderivative of $f(t)$ on $[a,b]_{\mathbb{T}}$, then we say $f$ is $CN$-delta integrable on $[a,b]_{\mathbb{T}}$ and we define

$$CN \int_a^b f(t)\Delta t := F(b) - F(a).$$

It follows from Theorem 2.3 that every $CN$-delta integrable function $f(t)$ on $[a,b]_{\mathbb{T}}$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and

$$HK \int_a^b f(t)\Delta t = CN \int_a^b f(t)\Delta t.$$

Hence the class of $HK$-delta integrable functions on $[a,b]_{\mathbb{T}}$ contains the class of Riemann delta integrable functions on $[a,b]_{\mathbb{T}}$ and the class of $CN$-delta integrable functions on $[a,b]_{\mathbb{T}}$. If $\mathbb{T} := [0,1] \cup [2,3]$, and $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) := 0$, $t \neq 1$ and $f(1) := 1$, then $f$ is a simple example of a function which is Riemann delta integrable on $[0,2]_{\mathbb{T}}$ (with $\int_a^b f(t)\Delta t = 1$), but is not $CN$-delta integrable on $[0,2]_{\mathbb{T}}$. For the next example let $\mathbb{T} := \{t_{2n} = \frac{1}{n} : n \in \mathbb{N}\} \cup \{t_{2n+1} = \frac{1}{n} - \frac{1}{n^3} : n = 2, 3, 4, \cdots\} \cup \{0\}$ and let $F : \mathbb{T} \to \mathbb{R}$ be defined by $F(t_{2n}) := t_{2n}^2, n \in \mathbb{N}$ $F(t_{2n+1}) = 0 = F(0)$, $n \geq 2$ and note that $F^{\Delta}(0) = 0$. Let $f(t) := F^{\Delta}(t), t \in [0,1)_{\mathbb{T}}$, $f(1) = 1$, then $f(t)$ is $CN$-delta integrable on $[0,1]_{\mathbb{T}}$ with $CN \int_0^1 f(t)\Delta t = F(1) - F(0) = 1$, but since, for $n \geq 3$,

$$f(t_{2n+1}) = F^{\Delta}(t_{2n+1}) = \frac{F(t_{2n}) - F(t_{2n+1})}{t_{2n} - t_{2n+1}} = n$$

we have $f(t)$ is not bounded on $[0,1]_{\mathbb{T}}$, so $f(t)$ is not Riemann delta integrable on $[0,1]_{\mathbb{T}}$.

Similar to the proof of Theorem 2.3 one can prove the following theorem.

**Theorem 2.4** *Assume $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ is continuous and there is a set $D \subset [a,b]_{\mathbb{T}}$ such that $F^{\nabla}(t) = f(t)$ for $t \in D$ and $[a,b]_{\mathbb{T}} \backslash D$ is countable and contains no left-scattered, right-dense points, then $f$ is $HK$–nabla integrable on $[a,b]_{\mathbb{T}}$ with*

$$HK \int_a^b f(t)\nabla t = F(b) - F(a).$$

For each of the remaining examples and theorems concerning Henstock–Kurzweil delta integration there are the corresponding results for Henstock–Kurzweil nabla integration which we won't bother to state. We next give an example of a function which is not delta integrable on $[0,1]_{\mathbb{T}}$, but is $HK$–delta integrable on $[0,1]_{\mathbb{T}}$, but is not absolutely $HK$-delta integrable on $[0,1]_{\mathbb{T}}$.

**Example 2.5** *Let $\mathbb{T} = \{t = \frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and define $f : \mathbb{T} \to \mathbb{R}$ by*

$$f(t) := \begin{cases} (-1)^n n, & t = \frac{1}{n} \\ L, & t = 0, \end{cases}$$

*where $L$ is any constant. Note that $f$ is not delta integrable on $[0,1]_{\mathbb{T}}$. Then it can be shown that if $F : \mathbb{T} \to \mathbb{R}$ is defined by*

$$F(t) := \begin{cases} 0, \quad t = 1 \\ \sum_{k=2}^{n} \frac{(-1)^{k+1}}{k-1}, & t = \frac{1}{n} \\ -\ln 2, & t = 0, \end{cases}$$

*then $F^{\Delta}(t) = f(t)$ for $t \in (0,1)_{\mathbb{T}}$ and $F$ is continuous $[0,1]_{\mathbb{T}}$ ( note the value of $F(0)$ is determined so that $F$ is continuous at 0). It follows by Theorem 2.3 with $D := (0,1)_{\mathbb{T}}$, that $f$ is $HK$-delta integrable on $[0,1]_{\mathbb{T}}$ and*

$$HK \int_0^1 f(t)\Delta t = F(1) - F(0) = \ln 2.$$

*However it can be shown that $HK \int_0^1 |f(t)|\Delta t$ does not exist ($f$ is not absolutely $HK$-delta integrable on $[0,1]_{\mathbb{T}}$).*

**Remark 2.6** *Lebesgue integration is an absolute integration. By this we mean $f$ is Lesbesgue integrable on $[a,b]$ iff $|f|$ is Lebesgue integrable on $[a,b]$. Example 2.5 shows that Henstock-Kurzweil delta integration does not have this weakness and is a nonabsolute integration. Indeed the Henstock-Kurtzweil delta integral is designed to integrate highly oscillatory functions.*

**Corollary 2.7** *If $f : \mathbb{T} \to \mathbb{R}$ is regulated and $a, b \in \mathbb{T}$, then $f$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$HK \int_a^b f(t)\Delta t = \int_a^b f(t)\Delta t.$$

*Moreover, if $G(t) := \int_a^t f(s)\Delta s$, then $G^{\Delta}(t) = f(t)$ except for a countable set.*

*Proof:* By [2, Theorem 1.70] we have, since $f$ is regulated, that there is a function $F : [a,b]_{\mathbb{T}} \to \mathbb{R}$ which is continuous on $[a,b]_{\mathbb{T}}$ and there is a set $D$ with $D \subset N_{\mu} \subset [a,b]_{\mathbb{T}}^{\kappa}$ such that $F^{\Delta}(t) = f(t)$ for $t \in D$ and $[a,b]_{\mathbb{T}} \backslash D$ is countable. Then using Theorem 2.3, $f$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ with

$$HK \int_a^b f(t)\Delta t = F(b) - F(a). \tag{2.10}$$

Since $f$ is regulated we have by [3, Theorem 5.21] that $f$ is delta integrable on $[a,b]_{\mathbb{T}}$, and then by [3, Theorem 5.39]

$$\int_a^b f(t)\Delta t = F(b) - F(a). \tag{2.11}$$

Hence from (2.10) and (2.11) we have the desired result

$$HK \int_a^b f(t)\Delta t = \int_a^b f(t)\Delta t.$$

□

Many of the simple properties of the $HK$-delta integral go through like the classical Henstock–Kurzweil integral where the following remarks are useful.

**Remark 2.8** *Let $\delta^1, \delta$ be $\Delta$-gauges for $[a,b]_{\mathbb{T}}$ such that $0 < \delta^1_L(t) \le \delta_L(t)$ for $t \in (a,b]_{\mathbb{T}}$ and $0 < \delta^1_R(t) \le \delta_R(t)$ for $t \in [a,b)_{\mathbb{T}}$ (write $\delta^1 \le \delta$ and we say $\delta^1$ is finer than $\delta$). If $\mathcal{P}_1$ is a $\delta^1$-fine partition of $[a,b]_{\mathbb{T}}$, then $\mathcal{P}_1$ is a $\delta$-fine partition of $[a,b]$.*

**Remark 2.9** *If $c \in [a,b]_{\mathbb{T}}$ and $\mathcal{P}$ is a $\delta$-fine partition, then there is a $\delta$-fine partition with $c$ as a tag point.*

**Remark 2.10** *If $c \in [a,b]_{\mathbb{T}}$ and $t_{i-1} \le c \le t_i$ is a tag point in a $\delta$-fine partition $\mathcal{P}$, then*

$$\mathcal{P}' := \{t_0 = a \le \xi_1 \le \cdots \le t_{i-1} \le c \le c \le c \le t_i \le \cdots \le t_n = b\}$$

*where $c$ is an end point and a tag point for the two intervals $[t_{i-1}, c]$ and $[c, t_i]$ is a $\delta$-fine partition and the Riemann sum corresponding to these two partitions is the same. This follows from the simple fact that*

$$f(c)[t_i - t_{i-1}] = f(c)[t_i - c] + f(c)[c - t_{i-1}].$$

**Remark 2.11** *Let $a < c < b$ be points in $\mathbb{T}$, then we may choose our gauge $\delta$ so that $\delta_R(t)$, $\delta_L(t) \le |t-c|$ for all $t \in \mathbb{T}\backslash\{c\}$. Then, if $\mathcal{P}$ is a $\delta$-fine partition of $[a,b]_{\mathbb{T}}$, then $\xi_{i_0} = c$ for some $i_0$. If $x_{i_0-1} < \xi_{i_0}$, then we may add $y_{i_0}$ to partition so that*

$$\{a = t_0 \le \xi_1 \le t_1 \le \cdots \le t_{i_0-1} \le \xi_{i_0} = y_{i_0} \le \cdots \le b\}$$

*so that*

$$\{a = t_0 \le \xi_1 \le t_1 \le \cdots \le t_{i_0-1} \le \xi_{i_0} = y_{i_0} = c\}$$

*and*

$$\{c = y_{i_0} \le \xi_0 \le \cdots \le t_n = b\}$$

*are $\delta$-fine partitions of $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$ respectively.*

Using these remarks and simple adjustments of the results in Lee Peng–Yee [8] one can prove the following theorem.

**Theorem 2.12** *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$. Then $f$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ iff $f$ is $HK$-delta integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$. Moreover, in this case*

$$HK\int_a^b f(t)\Delta t = HK\int_a^c f(t)\Delta t + HK\int_c^b f(t)\Delta t. \tag{2.12}$$

*Also if $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are $HK$-delta integrable on $[a,b]_{\mathbb{T}}$, then $\alpha f + \beta g$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$HK\int_a^b (\alpha f(t) + \beta g(t))\Delta t = \alpha\left(HK\int_a^b f(t)\Delta t\right) + \beta\left(HK\int_a^b f(t)\Delta t\right).$$

*Proof:* We will just show that if $f$ is $HK$-delta integrable on $[a,c]_{\mathbb{T}}$ and $[c,b]_{\mathbb{T}}$, then $f$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and (2.12) holds. Let

$$A := HK\int_a^c f(t)\Delta t, \quad B := HK\int_c^b f(t)\Delta t.$$

Let $\varepsilon > 0$ be given. Then there is a $\Delta$-gauge, $\delta^1$, of $[a,c]_{\mathbb{T}}$ and a $\Delta$-gauge, $\delta^2$, of $[c,b]_{\mathbb{T}}$ such that for all $\delta^1$-fine partitions $\mathcal{P}$ of $[a,c]_{\mathbb{T}}$ and all $\delta^2$-fine partitions $\mathcal{P}'$ of $[c,b]_{\mathbb{T}}$ we have that

$$|A - \sum_{i=1}^{n} f(\xi_i)(t_i - t_{i-1})| < \frac{\epsilon}{2}, \quad |B - \sum_{i=1}^{m} f(\xi_i')(t_i' - t_{i-1}')| < \frac{\epsilon}{2}. \tag{2.13}$$

We define a $\Delta$-gauge, $\delta = (\delta_L, \delta_R)$, on $[a,b]_{\mathbb{T}}$ by first defining $\delta_L$ as follows: $\delta_L(t) = \delta_L^1(t)$, $t \in [a,c)_{\mathbb{T}}$,

$$\delta_L(c) = \begin{cases} \delta_L^1(c), & \nu(c) = 0 \\ \min\{\delta_L^1(c), \frac{\nu(c)}{2}\}, & \nu(c) > 0, \end{cases}$$

$\delta_L(t) = \min\{\delta_L^2(t), \frac{t-c}{2}\}$, $t \in (c,b]_{\mathbb{T}}$, and then defining $\delta_R$ as follows: $\delta_R(t) = \delta_R^2(t)$, $t \in [c,b]_{\mathbb{T}}$,

$$\delta_R(t) = \min\{\delta_R^1(t), \max\{\mu(t), \frac{c-t}{2}\}\}, \quad t \in [a,c)_{\mathbb{T}}.$$

Now let $\mathcal{P}''$ be a $\delta$-fine partition of $[a,b]_{\mathbb{T}}$. Because of the way we defined $\delta$, there are two cases: either $c$ is a tag point for $\mathcal{P}''$, say $c = \xi_k''$, and $t_k'' > c$ or $\rho(c) < c$, $\rho(c)$ is a tag point for $\mathcal{P}''$, say $\rho(c) = \xi_k''$, and $t_k'' = c$. In the first case we have using

$$f(c)(t_k'' - t_{k-1}'') = f(c)(t_k'' - c) + f(c)(c - t_{k-1}'')$$

that

$$\begin{aligned} & |(A+B) - \sum_{i=1}^{p} f(\xi_i'')(t_i'' - t_i'')| \\ \leq\ & |A - \sum_{i=1}^{k-1} f(\xi'')(t_i'' - t_i') - f(c)(c - t_{k-1}'')| + |B - \sum_{i=k+1}^{p} f(\xi'')(t_i'' - t_i'') - f(c)(t_k'' - c)| \\ <\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

by (2.12), where we have used that the partition corresponding to the first term is finer than the partition $\mathcal{P}$ of $[a,c]_{\mathbb{T}}$ and the partition corresponding to the second term is finer than the partition $\mathcal{P}'$ of $[c,b]_{\mathbb{T}}$. The other case is easy and hence will be omited. $\square$

The proofs of the following two results are very similar to the proofs of Theorem 3.6 and Theorem 3.7 in [8] respectively and hence the proofs are omited.

**Theorem 2.13** *If $f$ and $g$ are $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$f(t) \leq g(t) \quad a.e.\ on\ [a,b)_{\mathbb{T}}$$

*then*

$$HK\int_a^b f(t)\Delta t \leq HK\int_a^b g(t)\Delta t.$$

**Theorem 2.14** *Assume $f$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$. Then given any $\varepsilon > 0$ there is a $\Delta$-gauge, $\delta$, of $[a,b]_{\mathbb{T}}$ such that*

$$\sum_{i=1}^{n} |HK \int_{t_{i-1}}^{t_i} f(t)\Delta t - f(\xi_i)(t_i - t_{i-1})| < \varepsilon$$

*for all $\delta$-fine partitions $\mathcal{P}$ of $[a,b]_{\mathbb{T}}$.*

**Definition 2.15** *We say a subset $S$ of a time scale $\mathbb{T}$ has delta measure zero provided $S$ contains no right-scattered points and $S$ has Lebesgue measure zero. We say a property $A$ holds delta almost everywhere (delta a.e.) on $\mathbb{T}$ provided there is a subset $S$ of $\mathbb{T}$ such that the property $A$ holds for all $t \in S$ and $S$ has delta measure zero.*

**Theorem 2.16 (Monotone Convergence Theorem)** *Let $f_k, f : [a,b]_{\mathbb{T}} \to \mathbb{T}$ and assume that*

*(i)* $f_k$ *is $HK$-delta integrable,* $k \in \mathbb{N}$;

*(ii)* $f_k \to f$ *delta ae in* $[a,b)_{\mathbb{T}}$;

*(iii)* $f_k \leq f_{n+1}$ *delta ae on* $[a,b)_{\mathbb{T}}$, $k \in \mathbb{N}$;

*(iv)* $\lim_{k\to\infty} HK \int_a^b f_k(t)\Delta t = I$.

*Then $f$ is $HK$-delta integrable on $[a,b]_{\mathbb{T}}$ and*

$$I = HK \int_a^b f(t)\Delta t.$$

*Proof:* Considering $f_k - f_1$ if necessary we can assume without loss of generality that

$$f_k(t) \geq 0 \quad \text{delta ae on} \quad [a,b)_{\mathbb{T}}.$$

Also to keep things easier we will just do the proof in the case where we replace (ii) by

$$f_k \to f \quad \text{for each} \quad t \in [a,b)_{\mathbb{T}} \tag{2.14}$$

and in place of (iii) we assume

$$f_k(t) \leq f_{k+1}(t), \quad t \in [a,b)_{\mathbb{T}}. \tag{2.15}$$

Let $\varepsilon > 0$ be given. Since $HK \int_a^b f_k(t)\Delta t$ is monotone nondecreasing with limit $I$, we can pick a positive integer $k_0$ so that

$$0 \leq I - HK \int_a^b f_k(t)\Delta t < \frac{\varepsilon}{3} \tag{2.16}$$

for all $k \geq k_0$. From (2.14) we have for each $t \in [a,b)_{\mathbb{T}}$ there is a positive integer $m(\varepsilon,t) \geq k_0$ such that

$$|f_{m(\varepsilon,t)}(t) - f(t)| < \frac{\varepsilon}{3(b-a)}. \tag{2.17}$$

Since each $f_k$ is HK-delta integrable on $[a,b]_{\mathbb{T}}$ we have from Theorem 2.14 there is a $\Delta$-gauge, $\delta^k$, of $[a,b]_{\mathbb{T}}$ such that

$$\sum |HK\int_u^v f_k(t)\Delta t - f_k(\xi)(v-u)| < \frac{\epsilon}{3\cdot 2^k} \tag{2.18}$$

for each $\delta^k$-fine partition of $[a,b]_{\mathbb{T}}$. Now define a $\Delta$-gauge, $\delta$, on $[a,b]_{\mathbb{T}}$ by

$$\delta(t) := \delta^{m(\varepsilon,t)}(t).$$

Let $\mathcal{P}$ be a $\delta$-fine partition, then using and (2.17) and (2.18)

$$\begin{aligned}
& |\sum_{i=1}^n f(\xi_i)(t_i - t_{i-1}) - I| \\
\leq\ & \sum_{i=1}^n |f(\xi_i) - f_{m(\varepsilon,\xi_i)}(\xi_i)|(t_i - t_{i-1}) \\
+\ & \sum_{i=1}^n |f_{m(\varepsilon,\xi_i)}(\xi_i)(t_i - t_{i-1}) - HK\int_{t_{i-1}}^{t_i} f_{m(\varepsilon,\xi_i)}(t)\Delta t| \\
+\ & |\sum_{i=1}^n HK\int_{t_{i-1}}^{t_i} f_{m(\varepsilon,\xi_i)}(t)\Delta t - I| \\
<\ & \frac{\varepsilon}{3(b-a)}(b-a) + \frac{\varepsilon}{3}\sum_{i=1}^\infty \frac{1}{2^i} + |\sum_{i=1}^n HK\int_{t_{i-1}}^{t_i} f_{m(\varepsilon,\xi_i)}(t)\Delta t - I| \\
=\ & \frac{2\varepsilon}{3} + |\sum_{i=1}^n HK\int_{t_{i-1}}^{t_i} f_{m(\varepsilon,\xi_i)}(t)\Delta t - I|
\end{aligned}$$

To complete the proof we need to show that the last term above is less than $\frac{\varepsilon}{3}$. To see this let

$$p := \min\{m(t,\xi_i) : 1 \leq i \leq n\} \geq k_0.$$

Then since

$$HK\int_{t_{i-1}}^{t_i} f_k(t)\Delta t$$

is monotone nondecreasing with respect to $k$ we have

$$\begin{aligned} HK\int_a^b f_p(t)\Delta t &= \sum_{i=1}^n HK\int_{t_{i-1}}^{t_i} f_p(t)\Delta t \\ &\leq \sum_{i=1}^n HK\int_{t_{i-1}}^{t_i} f_{m(\varepsilon,\xi_i)}(t)\Delta t \\ &\leq \sum_{i=1}^n HK\lim_{k\to\infty}\int_{t_{i-1}}^{t_i} f_k(t)\Delta t \\ &= \lim_{k\to\infty} HK\int_a^b f(t)\Delta t = I \end{aligned}$$

It follows from this that

$$I - \sum_{i=1}^n HK\int_{t_{i-1}}^{t_i} f_{m(\varepsilon,\xi_i)}(t)\Delta t \leq \frac{\varepsilon}{3}$$

and the proof is complete. □

The proofs of the remaining results in this section are now very similar to the proofs for the analogous results given in Lee Peng–Yee [8] and hence the proofs are omitted.

**Theorem 2.17 (Dominated Convergence Theorem)** *Assume*

*(i)* $f_k \to f$ *ae on* $[a,b]_{\mathbb{T}}$;

*(ii)* $g \leq f_k \leq h$ *ae on* $[a,b]_{\mathbb{T}}$;

*(iii)* $f_k$, $g$, $h$ *are* $HK$*-delta integrable on* $[a,b]$.

*Then* $f$ *is* $HK$*-delta integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$\lim_{k\to\infty} HK\int_a^b f_k(t)\Delta t = HK\int_a^b f(t)\Delta t.$$

**Definition 2.18** *We say* $f:[a,b]_{\mathbb{T}} \to \mathbb{R}$ *is absolutely* $HK$*-delta integrable on* $[a,b]_{\mathbb{T}}$ *provided* $f$ *and* $|f|$ *are* $HK$ *delta integrable on* $[a,b]_{\mathbb{T}}$.

**Theorem 2.19** *The function* $f$ *is Lebesgue delta integrable on* $[a,b]_{\mathbb{T}}$ *iff* $f$ *is absolutely* $HK$*-delta integrable on* $[a,b]_{\mathbb{T}}$.

**Theorem 2.20** *The function* $F$ *is absolutely continuous on* $[a,b]_{\mathbb{T}}$ *iff* $F^{\Delta}(t) = f(t)$ *delta ae on* $[a,b]_{\mathbb{T}}$ *is absolutely* $HK$*-delta integrable function* $f$ *on* $[a,b]_{\mathbb{T}}$. *Moreover*

$$HK\int_a^b |F^{\Delta}(t)|\Delta t = \int_{[a,b]_{\mathbb{T}}} |F^{\Delta}(t)|dt + \sum_{t_i\in\mathbb{N}_\mu} |F^{\sigma}(t_i) - F(t_i)|.$$

**Theorem 2.21** *If* $f(t) = 0$ *delta a.e. in* $[a,b]_{\mathbb{T}}$, *then* $f$ *is* $HK$*-delta integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$HK\int_a^b f(t)\Delta t = 0.$$

# 3 Applications

In this section we indicate that our results lead to a more general Taylor's theorem with remainder and a general mean value theorem. First we need the following lemma.

**Lemma 3.1 (Integration by Parts)** *Assume $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are differentiable and $f^{\Delta}$ and $g^{\Delta}$ are HK-delta integrable on $[a,b]_{\mathbb{T}}$, then*

$$HK\int_a^b f(t)g^{\Delta}(t)\Delta t = [f(t)g(t)]_a^b - HK\int_a^b f^{\Delta}(t)g(t)\Delta t.$$

*Proof:* This follows directly from the product rule

$$(f(t)g(t))^{\Delta} = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

and properties of $HK$-delta integrals. □

Before we state our next result we define as in Section 1.6 in [2] the so-called Taylor monomials $\{h_k(t,s)\}_{k=0}^{\infty}$ as follows:

$$h_0(t,s) = 1, \quad h_{n+1}(t,s) = \int_s^t h_n(t,\sigma(s))\Delta s, \quad t, s \in \mathbb{T}.$$

Note if $\mathbb{T} = \mathbb{R}$, then $h_n(t,s) = \frac{(t-s)^n}{n!}$ and if $\mathbb{T} = \mathbb{Z}$, then $h_n(t,s) = \frac{(t-s)^{\underline{n}}}{n!}$, where $(t-s)^{\underline{n}} := (t-s)(t-s-1)\cdots(t-s-n+1)$.

Now we can prove a very general Taylor's theorem with remainder (we do not need to assume $f^{\Delta^{n+1}}(t)$ is $rd$-continuous!).

**Theorem 3.2 (Taylor's Theorem)** *Assume $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$, is such that $f^{\Delta^{n+1}}(t)$ exists for $t \in \mathbb{T}^{\kappa^{n+1}}$ and $f^{\Delta^{n+1}}(t)$ is HK-delta integrable on $[a,b]_{\mathbb{T}}$. Then*

$$f(t) = \sum_{k=0}^{n} f^{\Delta^k}(t_0)h_k(t,t_0) + HK\int_{t_0}^t f^{\Delta^{n+1}}(s)h_{n+1}(t,\sigma(s))\Delta s$$

*where $t_0, t \in \mathbb{T}$.*

*Proof:* Integrating by parts we get that

$$\begin{aligned} HK\int_{t_0}^t f^{\Delta^{n+1}}(s)h_{n+1}(t,\sigma(s))\Delta s &= \left[f^{\Delta^n}(t)h_{n+1}(t,s)\right]_{s=t_0}^{s=t} - HK\int_{t_0}^t f^{\Delta^n}(s)h_{n+1}^{\Delta_s}(t,s)\Delta s \\ &= f^{\Delta^n}(t_0)h_{n+1}(t,t_0) - HK\int_{t_0}^t f^{\Delta^n}(s)h_{n+1}^{\Delta_s}(t,s)\Delta s. \end{aligned}$$

Simplifying and repeated integration by parts as in the proof of Theorem 1.113 in [2] leads to the desired result. □

In the next theorem we use the essential sup and the essential inf which we define as follows

$$essinf_{[a,b)}\ f(t) := \inf_N \inf_{t \notin N}\{f(t), N \subset [a,b)_{\mathbb{T}} \text{ has delta measure zero}\}$$

and

$$esssup_{[a,b]}\ f(t) := \sup_N \sup_{t\notin N}\{f(t), N\subset [a,b)_{\mathbb{T}} \text{ has delta measure zero}\}.$$

**Theorem 3.3 (Mean Value Theorem)** *Assume $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ and $f^\Delta$ is HK-integrable on $[a,b]_{\mathbb{T}}$, then*

$$essinf_{[a,b)} f^\Delta(t) \leq \frac{f(b)-f(a)}{\int_a^b (t-\sigma(s))\Delta s} \leq esssup_{[a,b)} f^\Delta(t).$$

*Proof:* By Taylor's theorem (Theorem 3.2) with $n = 0$,

$$\begin{aligned} f(b) &= f(a) + HK\int_a^b f^\Delta(s)h_1(t,\sigma(s))\Delta s \\ &= f(a) + HK\int_a^b f^\Delta(s)(t-\sigma(s))\Delta s. \end{aligned}$$

Hence

$$f(b) - f(a) = HK\int_a^b f^\Delta(s)(t-\sigma(s))\Delta s.$$

The result follows easily from this last equality. □

A special case of this mean value theorem is Corollary 1.68 in Bohner and Peterson [2].

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