A Wong-type oscillation theorem for second order linear dynamic equations on time scales

Jia Baoguo, Lynn Erbe, and Allan Peterson*
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE 68588-0130, U.S.A.
lerbe2@math.unl.edu, apeterso1@math.unl.edu,
Jia Baoguo
School of Mathematics and Computer Science
Zhongshan University
Guangzhou, China, 510275
mcsjbg@mail.sysu.edu.cn

Abstract. We obtain Wong-type oscillation theorems for second order linear dynamic equations on a time scale. The results obtained extend and are motivated by oscillation results due to Wong [**3**]. As a particular application of our results, we show that the difference equation

$$\Delta^2 x(n) + \frac{b(-1)^n}{n} x(n+1) = 0$$

is oscillatory iff $|b| > 1$.
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1. Introduction

In a fundamental paper [**3**], Wong extended and improved oscillation criteria due to many earlier authors for the differential equation

$$x'' + p(t)x = 0$$

in the cases when $p(t)$ is not eventually of one sign. His work also surveyed earlier results of Wintner [**15**], Fite [**9**], Hille [**11**], and Hartman [**10**] for the cases when $\int^\infty p(s)ds$ exists. In this paper we obtain a 'Wong-type' criterion for dynamic equations on time scales by means of a 'second-level Riccati equation' (see [**2**] for the discrete case) or what Wong refers to as a new Riccati integral equation in the continuous case. Using this approach, one is able to handle various critical cases. These ideas are of particular importance in treating the case when $P(t) := \int_t^\infty p(s)ds$ is not of one sign for large t.

For $\mathbb{T} = \mathbb{R}$, Wong [**3**, Theorem 2] proved the following theorem.

Theorem A: Assume $\int_0^\infty p(t)\,dt$ converges, $P(t) := \int_t^\infty p(s)\,ds$ is such that the improper integral $\int_0^\infty P(t)\,dt$ converges and

$$\int_0^\infty \exp\left\{-4\int_0^t \bar{P}(u)du\right\} dt < \infty;$$

where

$$\bar{P}(t) := \int_t^\infty P^2(s)Q_P(s,t)ds, \quad Q_P(s,t) := \exp\left\{2\int_t^s P(\tau)d\tau\right\}.$$

Then $x'' + p(t)x(t) = 0$ is oscillatory.

In [**14**] and [**3**], Willett and Wong consider the equation

$$x'' + \frac{b\sin\lambda t}{t}x(t) = 0 \tag{1.1}$$

and proved that (1.1) is oscillatory when $|\frac{b}{\lambda}| > \frac{1}{\sqrt{2}}$ and showed that $\frac{1}{\sqrt{2}}$ is a critical value, i.e., (1.1) is nonoscillatory when $|\frac{b}{\lambda}| \leq \frac{1}{\sqrt{2}}$. One can show that (1.1) is oscillatory using Wong's Theorem A, when $|\frac{b}{\lambda}| > \frac{1}{\sqrt{2}}$ (we leave these details to the interested reader). Willett [**14**] used the Riccati integral equation to establish oscillation in this case. Wong using the same integral equation was able to prove nonoscillation in the critical case $|\frac{b}{\lambda}| = \frac{1}{\sqrt{2}}$.

For completeness, (see [**5**] and [**6**] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup\mathbb{T} = \infty$. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf\varnothing = \sup\mathbb{T}$, where $\varnothing$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf\mathbb{T}$ we say t is left-dense. Given an interval $[c,d] := \{t \in \mathbb{T} : c \leq t \leq d\}$ in $\mathbb{T}$ the notation $[c,d]^\kappa$ denotes the interval $[c,d]$ in case $\rho(d) = d$ and denotes the interval $[c,d)$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$. We say that $x : \mathbb{T} \to \mathbb{R}$ is differentiable at $t \in \mathbb{T}$ provided

$$x^\Delta(t) := \lim_{s\to t}\frac{x(t) - x(s)}{t - s},$$

exists when $\sigma(t) = t$ (here by $s \to t$ it is understood that s approaches t in the time scale) and when x is continuous at t and $\sigma(t) > t$

$$x^{\Delta}(t) := \frac{x(\sigma(t)) - x(t)}{\mu(t)}.$$

Note that if $\mathbb{T} = \mathbb{R}$, then the delta derivaive is just the standard derivative, and when $\mathbb{T} = \mathbb{Z}$ the delta derivative is just the forward difference operator. Hence our results contain the discrete and continuous cases as special cases and generalize these results to arbitray time scales.

Our main concern in this paper is the second order dynamic equation

$$x^{\Delta\Delta} + p(t)x^{\sigma} = 0, \tag{1.2}$$

on time scale $\mathbb{T}$, where p is right-dense continuous functions on $\mathbb{T}$ and the improper integral $\int_{t_0}^{\infty} p(s)\Delta s$ converges.

In this paper, we extend Theorem A of Wong to dynamic equations on time scales and, as an application, we show that the difference equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n}x(n+1) = 0 \tag{1.3}$$

is oscillatory if $|b| > 1$. From [**1**], it follows that 1 is the critical value, i.e. (1.3) is nonoscillatory when $b \leq 1$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{rd}. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable on $[c, d]^{\kappa}$ and whose delta derivative is rd-continous on $[c, d]^{\kappa}$ is denoted by C^1_{rd}. (See [**5**] for additional properties of rd-continuous functions). If $F^{\Delta}(t) = f(t)$ for $t \in \mathbb{T}$, then $F(t)$ is said to be a (delta) antiderivative of $f(t)$. If $F(t)$ is a (delta) antiderivative of $f(t)$ then we define the Cauchy (delta) integral of $f(t)$ on $[a, b]$ by

$$\int_a^b f(t)\ \Delta t = F(b) - F(a).$$

See [**5**] for elementary properties of the Cauchy integral. One interesting result ([**5**, Theorem 1.74]) is that if $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$ then it has a (delta) antiderivative on $\mathbb{T}$ and hence the integral $\int_a^b f(t)\ \Delta t$ exists.

We recall that a solution of equation (1.2) is said to be oscillatory on $[a, \infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Equation (1.2) is said to be oscillatory in case all of its solutions are oscillatory.

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided that

$$1 + \mu(t)p(t) \neq 0, \ \ t \in \mathbb{T}$$

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are right-dense continuous and regressive by $\mathcal{R}$. If $p \in \mathcal{R}$, then we can define (see [**5**, Definition 2.29]) the (generalized) exponential function by

$$e_p(t,s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\}$$

for $t \in \mathbb{T}, s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{Log(1+hz)}{h} & \text{if } h \neq 0 \\ z & \text{h=0,} \end{cases}$$

where *Log* denotes the principal logarithm function. We also define the set of positively regressive functions to be all $p \in \mathcal{R}$ such that $1 + \mu(t)p(t) > 0$, $t \in \mathbb{T}$. An important fact (see [**5**, Theorem 2.44]) that we use thoughout this paper is that if $p \in \mathcal{R}^+$, then $e_p(t, t_0) > 0$, $t \in \mathbb{T}$.

2. Some definitions and Lemmas

Let $\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following:(see [**7**]).

Condition (C) We say that $\mathbb{T}$ satisfies condition C if there is an $M > 0$ such that

$$\chi(t) \le M\mu(t), \quad t \in \mathbb{T}.$$

LEMMA 2.1. *Assume that $\mathbb{T}$ satisfies condition (C), assume that equation* (1.2) *is nonoscillatory, and let $x(t)$ be a solution of* (1.2) *with $x(t) > 0$ on $[t_0, \infty)$. Then*

$$z(t) := \frac{x^\Delta(t)}{x(t)}$$

is a solution of the Riccati equation

$$z^\Delta + p(t) + \frac{z^2}{1 + \mu(t)z} = 0.$$

on $[t_0, \infty)$. Moreover, if $\int_{t_0}^\infty p(t)\Delta t$ is convergent, then $\int_{t_0}^\infty \frac{z^2(s)}{1+\mu(s)z(s)}\Delta s$ is also convergent and $\lim_{t\to\infty} z(t) = 0$.

PROOF. Assume that (1.2) is nonoscillatory and let $x(t) > 0$ be a solution of (1.2). We can assume without loss of generality that $\sup \hat{\mathbb{T}} = \infty$, since otherwise $\mathbb{T}$ is eventually a real interval and so the result reduces to a well-known result of Hartman [**10**]. From the Riccati equation we have

$$z(t) + \int_{t_0}^t p(s)\Delta s + \int_{t_0}^t \frac{z^2(s)}{1 + \mu(s)z(s)}\Delta s = z(t_0). \tag{2.1}$$

Since $\sup \hat{\mathbb{T}} = \infty$, there exists a sequence $\{t_k\} \subset \mathbb{T}$ with $t_k \to \infty$ and $\mu(t_k) > 0$ such that $1 \leq M\mu(t_k)$. Therefore, $z(t_k) > -\frac{1}{\mu(t_k)} \geq -M$. From (2.1), we get

$$-M + \int_{t_0}^{t_k} p(s)\Delta s + \int_{t_0}^{t_k} \frac{z^2(s)}{1+\mu(s)z(s)}\Delta s < z(t_0).$$

Hence, if $\lim_{t\to\infty} \int_{t_0}^{t} \frac{z^2(s)}{1+\mu(s)z(s)}\Delta s = +\infty$, then we have $\lim_{k\to} \int_{t_0}^{t_k} p(s)\Delta s = -\infty$, contradicting our assumption. Consequently, it follows that

$$\int_{t_0}^{\infty} \frac{z^2(s)}{1+\mu(s)z(s)}\Delta s$$

is convergent.

Let $R(t) := \frac{z^2(t)}{1+\mu(t)z(t)}$. Note that $\int_t^{\sigma(t)} g(s)\Delta s = \mu(t)g(t)$. Then, for n_0 sufficiently large, we have

$$\sum_{k=n_0}^{\infty} \mu(t_k)R(t_k) = \sum_{k=n_0}^{\infty} \int_{t_k}^{\sigma(t_k)} R(t)\Delta t \leq \int_{t_0}^{\infty} R(s)\Delta s < +\infty. \tag{2.2}$$

From (2.1), we get that $\lim_{t\to\infty} z(t)$ exists. Assuming that $\lim_{t\to\infty} z(t) = B \neq 0$. Then

$$\frac{|B|}{2} < |z(t)| < 2|B|, \quad \text{for large } t. \tag{2.3}$$

So from (2.2) and (2.3), we get that

$$\begin{aligned} 0 = \lim_{k\to\infty} |\mu(t_k)R(t_k)| &= \lim_{k\to\infty} |\frac{\mu(t_k)z^2(t_k)}{1+\mu(t_k)z(t_k)}| \\ &= \lim_{k\to\infty} \frac{z^2(t_k)}{|\frac{1}{\mu(t_k)} + z(t_k)|} > \frac{(\frac{|B|}{2})^2}{M+2|B|} > 0, \end{aligned}$$

which is a contradiction. So $\lim_{t\to\infty} z(t) = 0$. □

We will need below conditions which guarantee that $\int_1^t \frac{1}{s}\Delta s$ does not grow faster than $M \ln t$, for some $M > 0$. For a time scale $\mathbb{T}$, the following example shows that the inequality $\int_1^t \frac{1}{s}\Delta s \leq M \ln t$, for any $M > 1$, does not always hold in general without some additional restrictions.

Example 2.2. Consider the time scale

$$\mathbb{T} = \{2^{2^k}, k \in \mathbb{N}_0\}.$$

It is easy to see from the definition that for $t_k = 2^{2^k}$ we have

$$\lim_{k\to\infty} \frac{\int_{t_0}^{t_k} \frac{1}{s}\Delta s}{\ln t_k} = \frac{1}{\ln 2} \lim_{k\to\infty} \frac{1}{2^k} \sum_{j=0}^{k-1} \left(2^{2^j} - 1\right) = \infty.$$

So we shall impose an additional assumption on the time scale $\mathbb{T}$. We note first that if $\mathbb{T}$ satisfies condition (C), then the set

$$\check{\mathbb{T}} = \{t \in \mathbb{T} |\ t > 0 \text{ is isolated or right-scattered or left-scattered}\}$$

is necessarily countable. We introduce the following

Condition (D) Suppose that $\mathbb{T}$ satisfies condition (C) and let

$$\check{\mathbb{T}} = \{t_0, t_1, t_2, \cdots, t_k, \cdots\},$$

where

$$0 < t_0 < t_1 < t_2 < \cdots < t_k < \cdots .$$

Then we say $\mathbb{T}$ satisfies **Condition (D)** if there is a constant $K > 1$ such that

$$\frac{t_{k+1} - t_k}{t_k - t_{k-1}} \le K, \quad \text{for all } k \ge 1.$$

We will use the following lemma to prove Theorem 3.1 which is one of our main results.

LEMMA 2.3. *Assume that $\mathbb{T}$ satisfies condition (D) and suppose that $x(t)$ is a solution of* (1.2) *that satisfies $x(t) > 0$ for $t \ge T = t_k$, for some $k \ge 0$. Then we have, for $t \in \mathbb{T}$, $t \ge T$,*

$$\ln \frac{x(t)}{x(T)} \le \int_T^t \frac{x^\Delta(s)}{x(s)}\, \Delta s \quad \textit{and} \quad \int_T^t \frac{1}{s}\, \Delta s \le K \ln \frac{t}{T}.$$

PROOF. We will use below the inequality $x-1 > \ln x$, for $x > 1$. Assume $x(t)$ is a solution of (1.2) with $x(t) > 0$ on $[T, \infty)$, where $T = t_k$ for some $k \ge 1$. First we claim for any $i \ge k$

$$\int_{t_i}^{t_{i+1}} \frac{x^\Delta(t)}{x(t)} \Delta t \ge \ln \frac{x(t_{i+1})}{x(t_i)} \quad \text{and} \quad \int_{t_i}^{t_{i+1}} \frac{1}{t} \Delta t \le K \frac{t_i - t_{i-1}}{t_i}.$$

First we show this for the first case when the real interval $[t_i, t_{i+1}] \subset \mathbb{T}$. In this case

$$\int_{t_i}^{t_{i+1}} \frac{x^\Delta(t)}{x(t)} \Delta t = \ln \frac{x(t_{i+1})}{x(t_i)},$$

$$\int_{t_i}^{t_{i+1}} \frac{1}{t} \Delta t = \ln \frac{t_{i+1}}{t_i} \le \frac{t_{i+1} - t_i}{t_i} \le K \frac{t_i - t_{i-1}}{t_i}.$$

by Condition (D).

Next consider the case where $\sigma(t_i) = t_{i+1}$. In this case

$$\int_{t_i}^{t_{i+1}} \frac{x^\Delta(t)}{x(t)} \Delta t = \frac{x^\Delta(t_i)}{x(t_i)} \mu(t_i) = \frac{x(t_{i+1})}{x(t_i)} - 1 \ge \ln \frac{x(t_{i+1})}{x(t_i)},$$

and

$$\int_{t_i}^{t_{i+1}} \frac{1}{t} \Delta t = \frac{t_{i+1} - t_i}{t_i} \le K \frac{t_i - t_{i-1}}{t_i}.$$

This completes the proof of our claim, which we now use below. Let $t \in \mathbb{T}$, with $t \geq T$. Then there is an $i \geq k$ such that $t_i \leq t < t_{i+1}$. Therefore, we have

$$\int_T^t \frac{x^\Delta(s)}{x(s)}\Delta s = \sum_{j=k}^{i-1}\int_{t_j}^{t_{j+1}} \frac{x^\Delta(s)}{x(s)}\Delta s + \int_{t_i}^t \frac{x^\Delta(s)}{x(s)}\Delta s$$

$$\geq \sum_{j=k}^{i-1} \ln\frac{x(t_{j+1})}{x(t_j)} + \ln\frac{x(t)}{x(t_i)} = \ln\frac{x(t)}{x(T)},$$

and

$$\int_T^t \frac{1}{s}\,\Delta s = \sum_{j=k}^{i-1}\int_{t_j}^{t_{j+1}} \frac{1}{s}\,\Delta s + \int_{t_i}^t \frac{1}{s}\,\Delta s$$

$$\leq K\sum_{j=k}^{i-1}\frac{t_j - t_{j-1}}{t_j} + \int_{t_i}^t \frac{1}{s}\,\Delta s \leq K\int_T^t \frac{1}{s}\,ds \leq K\ln\frac{t}{T}.$$

This completes the proof of the lemma. □

The following lemma appears in [**5**].

Lemma 2.4. *Assume $a \in \mathbb{T}$, and equation (1.2) is nonoscillatory on $[a,\infty)$, then there is a solution u, called a recessive solution at ∞, such that for any second linearly independent solution v, called a dominant solution at ∞, we have*

(2.4)
$$\lim_{t\to\infty}\frac{u(t)}{v(t)} = 0, \quad \int_b^\infty \frac{1}{r(t)u(t)u^\sigma(t)}\Delta t = \infty \ \text{ and } \int_b^\infty \frac{1}{r(t)v(t)v^\sigma(t)}\Delta t < \infty,$$

where $b > a$ is sufficiently large. Furthermore,

$$\frac{r(t)v^\Delta(t)}{v(t)} > \frac{r(t)u^\Delta(t)}{u(t)} \tag{2.5}$$

for t sufficiently large.

Lemma 2.5. *Assume $\int_{t_0}^\infty p(t)\Delta t$ converges and let $P(t) = \int_t^\infty p(s)\Delta s$. Assume further that $\mu(t)$ is bounded. Then for large enough T, the functions $-P$, P, and $Q := \frac{-4P}{(1+\mu P)^2} \in \mathcal{R}^+$ and*

$$e_Q(t,T) = e_{\frac{-4P}{(1+\mu P)^2}}(t,T) = \left[\frac{e_{-P}(t,T)}{e_P(t,T)}\right]^2. \tag{2.6}$$

Furthermore $y(t) = e_Q(t,T)$ satisfies the dynamic equation

$$y^\Delta + P[y + y^\sigma] + 2P\sqrt{y}\sqrt{y^\sigma} = 0$$

Proof. Note that, since $\mu(t)$ is bounded we have

$$\lim_{t\to\infty}(1+\mu(t)P(t)) = \lim_{t\to\infty}(1-\mu(t)P(t)) = 1$$

Hence for $T \in [t_0, \infty)$ sufficiently large

$$1 + \mu(t)P(t) > 0, \ \ 1 - \mu(t)P(t) > 0 \tag{2.7}$$

on $[T, \infty)_{\mathbb{T}}$. Hence P, $-P$ are rd-continuous and positively regressive and so the generalized exponential functions exist and are positive on $[T, \infty)_{\mathbb{T}}$. Moreover, we have

$$1 + \mu(t)\left[\frac{-4P(t)}{(1+\mu(t)P(t))^2}\right] = \left[\frac{1-\mu(t)P(t)}{1+\mu(t)P(t)}\right]^2 > 0 \tag{2.8}$$

on $[T, \infty)_{\mathbb{T}}$. Therefore, since $\frac{-4P(t)}{(1+\mu(t)P(t))^2}$ is rd-continuous on $[T, \infty)_{\mathbb{T}}$, the exponential function $e_Q(t, T) = e_{\frac{-4P}{(1+\mu P)^2}}(t, T)$ exists and is positive on $[T, \infty)_{\mathbb{T}}$. To see that (2.6) holds, from the properties of the $\odot$ and $\ominus$ operations (see [**5**, Theorem 2.36] and [**6**, Theorem 2.44]), we have that

$$\begin{aligned} \left[\frac{e_{-P}(t,T)}{e_P(t,T)}\right]^2 &= e^2_{-P\ominus P}(t,T) = e^2_{\frac{-2P}{(1+\mu P)}}(t,T) \\ &= e_{2\odot\frac{-2P}{(1+\mu P)}}(t,T) = e_{\frac{-4P}{(1+\mu P)^2}}(t,T) = e_Q(t,T). \end{aligned}$$

□

3. A Hartman-type oscillation theorem for dynamic equations

The following Hartman-type theorem for the dynamic equation will be used to prove Theorem 4.1 which may be regarded as a Wong-type oscillation result.

THEOREM 3.1. *Assume that $\mathbb{T}$ satisfies condition (D) and that $\mu(t)$ is bounded. Assume $\int_{t_0}^{\infty} p(t)\Delta t$ is convergent and that*

$$\int_T^{\infty} e_Q(t, T)\Delta t < \infty, \tag{3.1}$$

where $P(t) = \int_t^{\infty} p(s)\Delta s$ and $Q = \frac{-4P}{(1+\mu P)^2}$ for T sufficiently large. Then (1.2) is oscillatory on $[t_0, \infty)$

Before we prove this result we make the following remark.

REMARK 3.2. In the original result of Hartman [**12**], it was shown that

$$\int_T^{\infty} \exp\left(-4\int_T^t P(s)ds\right) dt < \infty$$

implies oscillation. Theorem A of Wong extends this by replacing $P(t)$ by $\bar{P}(t)$.

PROOF. Assume (1.2) is nonoscillatory on $[t_0, \infty)$. Then there is a $T \in [t_0, \infty)$ such that (by Lemma 2.5)

$$-P, \quad P, \quad \frac{-4P}{(1+\mu P)^2} \in \mathcal{R}^+$$

and there is a dominant solution $x(t)$ of (1.2), with $x(t) > 0$ on $[T, \infty)$. Make the Riccati transformation $z(t) = \frac{x^\Delta(t)}{x(t)}$. Then by Lemma 2.1, $z(t)$ is a solution of the Riccati equation

$$z^\Delta + p(t) + \frac{z^2}{1+\mu(t)z} = 0$$

on $[T, \infty)$ with

$$1 + \mu(t)z(t) > 0, \quad t \in [T, \infty).$$

By Lemma 2.1, integrating the Riccati equation from t to ∞, we get that

$$z(t) = \int_t^\infty p(s)\Delta s + \int_t^\infty R(s)\Delta s,$$

where

$$R(s) = \frac{z^2(s)}{1+\mu(s)z(s)}.$$

First we show that

$$w(t) := \int_t^\infty R(s)\Delta s = z(t) - P(t)$$

satisfies the dynamic equation

$$Lw(t) := w^\Delta(t) + w(t)w(\sigma(t)) + P(t)[w(t) + w(\sigma(t))] + P^2(t) = 0 \tag{3.2}$$

To see this, consider (note that $w^\Delta(t) = -R(t)$)

$$\begin{aligned}
& w^\Delta(t) + w(t)w(\sigma(t)) + P(t)[w(t) + w(\sigma(t))] + P^2(t) \\
= \;& -R(t) + [w(t) + P(t)]w^\sigma(t) + P(t)w(t) + P^2(t) \\
= \;& -R(t) + z(t)w^\sigma(t) + P(t)[w(t) + P(t)] \\
= \;& -R(t) + z(t)[w(t) - \mu(t)R(t)] + P(t)z(t) \\
= \;& -R(t) + z(t)[w(t) + P(t)] - \mu(t)R(t)z(t) \\
= \;& -R(t) + z^2(t) - \mu(t)R(t)z(t) \\
= \;& z^2(t) - R(t)[1 + \mu(t)z(t)] \\
= \;& z^2(t) - z^2(t) \\
= \;& 0.
\end{aligned}$$

Let

$$y(t) := w(T)e_{\frac{-4P}{(1+\mu P)2}}(t, T) = w(T)e_Q(t, T).$$

We now show that $Ly(t) \geq 0$ on $[T, \infty)$. Using the inequality $a^2 + b^2 \geq 2|a||b|$ and Lemma 2.5 we get

$$\begin{aligned}
& y^\Delta(t) + y(t)y(\sigma(t)) + P(t)[y(t) + y(\sigma(t))] + P^2(t) \\
\geq \;& y^\Delta(t) + P(t)[y(t) + y(\sigma(t))] + 2P(t)y(t)y^\sigma(t) \\
= \;& 0, \quad t \in [T, \infty).
\end{aligned}$$

That is, we have that

$$Lw(t) = 0, \quad Ly(t) \geq 0 \quad \text{on} \quad [T, \infty), \quad \text{and} \quad y(T) = w(T).$$

We now use time scale induction on $[T, \infty)$ ([**5**, Theorem 1.7]) to prove that

$$y(t) \geq w(t), \quad t \in [T, \infty).$$

Since $y(T) = w(T)$ we have that part (i) of [**5**, Theorem 1.7] holds. Next assume that $t \in [T, \infty)$, $\sigma(t) \geq t$, and $y(t) \geq w(t)$. To prove part (ii) of [**5**, Theorem 1.7] we now show that $y(\sigma(t)) \geq w(\sigma(t))$. To see this note that solving $Lw(t) = 0$ for $w(\sigma(t))$ and similarly solving $Ly(t) \geq 0$ for $y(\sigma(t))$ we get (note that $1 + \mu(t)P(t) + \mu(t)w(t) > 0$ for large t)

$$w(\sigma(t)) = \frac{w(t) - \mu(t)P(t)w(t) - \mu(t)P^2(t)}{1 + \mu(t)P(t) + \mu(t)w(t)} \tag{3.3}$$

$$y(\sigma(t)) \geq \frac{y(t) - \mu(t)P(t)y(t) - \mu(t)P^2(t)}{1 + \mu(t)y(t) + \mu(t)P(t)}. \tag{3.4}$$

Letting

$$g(x, t) := \frac{[1 - \mu(t)P(t)]x - \mu(t)P^2(t)}{1 + \mu(t)x + \mu(t)P(t)}$$

we get

$$\frac{\partial}{\partial x} g(x, t) := \frac{1}{(1 + \mu(t)x + \mu(t)P(t))^2} > 0$$

for $t \in [T, \infty)$, $x > 0$. Hence it follows from (3.3), (3.4) and the induction assumption $y(t) \geq w(t)$ that $y(\sigma(t)) \geq w(\sigma(t))$. Hence part (ii) of [**5**, Theorem 1.7] holds. To prove part (iii) of the induction, assume $t_1 \in [T, \infty)$ $\sigma(t_1) = t_1$ and $y(t_1) \geq w(t_1)$. Because of our assumptions on the type of time scales we are considering (Condition (D)), there is an $\epsilon > 0$, sufficiently small, so that the real interval $[t_1, t_1 + \epsilon] \subset [T, \infty)$. Now on the interval $[t_1, t_1 + \epsilon]$ we get that $w(t)$ and $y(t)$ solve

$$w' = -w^2 - 2P(t)w - P^2(t), \quad y' \geq -y^2 - 2P(t)y - P^2(t)$$

Since by the induction assumption $y(t_1) \geq w(t_1)$, we get from [**10**, Theorem 4.1] that

$$y(t) \geq w(t), \quad t \in [T, \infty).$$

Since part (iv) [**5**, Theorem 1.7] is trivially true our induction is complete and we have

$$y(t) \geq w(t), \quad t \in [T, \infty).$$

Hence

$$\int_T^\infty w(t)\, \Delta t \leq \int_T^\infty y(t)\, \Delta t = w(T) \int_T^\infty e_Q(t, T) \Delta t < \infty$$

by (3.1). Using integration by parts and $w(t) = \int_t^\infty R(s)\,\Delta s$ we get

$$\begin{aligned} \int_T^t w(t)\,\Delta t &= (t-T)w(t) + \int_T^t (\sigma(s)-T)R(s)\Delta s \\ &\geq \int_T^t (\sigma(s)-T)R(s)\Delta s. \end{aligned}$$

It follows that

$$\int_T^\infty (s-T)R(s)\Delta s \leq \int_T^\infty (\sigma(s)-T)R(s)\Delta s \leq \int_T^\infty w(t)\,\Delta t < \infty,$$

and consequently, we have

$$\int_T^\infty sR(s)\Delta s < \infty.$$

Therefore, since $\mu(t)$ is bounded and $\lim_{t\to\infty} z(t) = 0$, it follows that

$$C := \int_T^\infty sz^2(s)\,\Delta s < \infty.$$

Using the Cauchy–Schwarz inequality we get

$$\left(\int_T^t z(s)\,\Delta s\right)^2 \leq \int_T^t \frac{1}{s}\Delta s \int_T^t sz^2(s)\,\Delta s \leq C\int_T^t \frac{1}{s}\Delta s.$$

It then follows from Lemma 2.3 that there is a constant $M > 0$ such that

$$\ln\left(\frac{x(t)}{x(T)}\right) \leq M\left(\ln\frac{t}{T}\right)^{\frac{1}{2}}, \quad t \in [T,\infty).$$

Pick $T_1 \in (T,\infty)$ so that $M(\ln\frac{t}{T})^{\frac{1}{2}} \leq \frac{1}{2}\ln t$, $t \in [T_1,\infty)$. Then we get the inequality

$$\ln\left(\frac{x(t)}{x(T)}\right) \leq \frac{1}{2}\ln t, \quad t \in [T_1,\infty).$$

Thus

$$\frac{1}{x(t)} \geq \frac{1}{x(T)t^{\frac{1}{2}}} \quad \text{and} \quad \frac{1}{x(\sigma(t))} \geq \frac{1}{x(T)(\sigma(t))^{\frac{1}{2}}},$$

for $t \in [T_1,\infty)$.

Since $\mu(t)$ is bounded, there exists $M_1 > 0$ such that

$$\int_{T_1}^\infty \frac{1}{x(t)x(\sigma(t))}\Delta t \geq \frac{1}{x^2(T)}\int_{T_1}^\infty \frac{1}{t^{\frac{1}{2}}(t+M_1)^{\frac{1}{2}}}\Delta t = \infty,$$

which contradicts the fact that $x(t)$ is a dominant solution (see Lemma 2.4 and equation (2.4)). This completes the proof. □

4. Wong-type oscillation and nonoscillation theorem for dynamic equations

Our first result is a necessary condition for nonoscillation under the assumptions that $\mathbb{T}$ satisfies condition (D) and that $\mu(t)$ is bounded.

THEOREM 4.1. *Assume that* $\int_{t_0}^{\infty} p(t)\Delta t$ *is convergent,* $\mu(t)$ *is bounded and that* $\mathbb{T}$ *satisfies condition* (D). *If* (1.2) *is nonoscillatory, then there is a* $T \in [t_0, \infty)$ *such that*

$$\int_T^{\infty} P^2(t) e_{\frac{2P}{1-\mu P}}(t, T)\Delta t < \infty$$

where $P(t) := \int_t^{\infty} p(s)\Delta s$. *Also, if* $x(t)$ *is a positive solution of* (1.2) *on* $[T, \infty)$ *and* $z(t) := \frac{x^{\Delta}(t)}{x(t)}$, *then* $z(t)$ *is a solution of the Riccati equation*

$$z^{\Delta} + p(t) + \frac{z^2}{1 + \mu(t)z} = 0.$$

on $[T, \infty)$, *with* $1 + \mu(t)z(t) > 0$ *on* $[T, \infty)$. *Furthermore,*

$$w(t) := \int_t^{\infty} \frac{z^2(s)}{1 + \mu(t)z(s)}\Delta s > 0$$

satisfies the integral equation

$$w(t) = \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s, t)\frac{P^2(s) + w(s)w(\sigma(s))}{1 - \mu(s)P(s)}\Delta s. \tag{4.1}$$

for $t \in [T, \infty)$.

PROOF. Assume (1.2) is nonoscillatory, then by Lemma 2.1 there is a $T \in [t_0, \infty)$ and a positive dominant solution $x(t)$ of (1.2) on $[T, \infty)$ such that (here we assume that T is chosen so that $1 \pm \mu(t)P(t) > 0$ on $[T, \infty)$) $z(t) = \frac{x^{\Delta}(t)}{x(t)}$ satisfies the conclusions in Theorem 4.1. Integrating the Riccati equation from t $(t \geq T)$ to ∞, we get that

$$z(t) = \int_t^{\infty} p(s)\Delta s + \int_t^{\infty} R(s)\Delta s,$$

where $R(s) = \frac{z^2(s)}{1+\mu(s)z(s)}$.

Let us define $w(t) := \int_t^{\infty} R(s)\Delta s > 0$. Then we have as in Theorem 3.1 that $w(t) := z(t) - P(t)$ satisfies (3.2) for $t \in [T, \infty)$. From (3.2) we get that $w(t)$ solves the dynamic equation

$$w^{\Delta} + 2wP + \mu P w^{\Delta} = -[ww^{\sigma} + P^2].$$

Solving for w^{Δ} we get

$$w^{\Delta} = \frac{-2P}{1 + \mu P}w - \frac{ww^{\sigma} + P^2}{1 + \mu P}.$$

Now by the variation of constants formula [**5**, Page 78] we get

$$w(t)=e_{\frac{-2P}{1+\mu P}}(t,T)w(T)-e_{\frac{-2P}{1+\mu P}}(t,T)\int_T^t e_{\frac{-2P}{1+\mu P}}(T,\sigma(s))\frac{w(s)w^\sigma(s)+P^2(s)}{1+\mu(s)P(s)}\Delta s.$$

Since

$$e_{\frac{-2P}{1+\mu P}}(T,\sigma(s))=e_{\frac{2P}{1-\mu P}}(s,T)\,\frac{1+\mu(s)P(s)}{1-\mu(s)P(s)},$$

this can be rewritten in the form

$$w(t)=e_{\frac{-2P}{1+\mu P}}(t,T)w(T)-e_{\frac{-2P}{1+\mu P}}(t,T)\int_T^t e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w^\sigma(s)+P^2(s)}{1-\mu(s)P(s)}\Delta s.$$

Multiplying both sides by $e_{\frac{2P}{1-\mu P}}(t,T)$ we get

$$e_{\frac{2P}{1-\mu P}}(t,T)w(t)=w(T)-\int_T^t e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w^\sigma(s)+P^2(s)}{1-\mu(s)P(s)}\Delta s. \tag{4.2}$$

It follows from this that $e_{\frac{2P}{1-\mu P}}(t,T)w(t)$ is decreasing and since it is positive we get

$$A:=\lim_{t\to\infty}e_{\frac{2P}{1-\mu P}}(t,T)w(t)\geq 0.$$

We claim that $A=0$. Assume not, then $A>0$. We show that this leads to a contradiction. Letting $t\to\infty$ in (4.2) we have

$$\begin{aligned} A &= w(T)-\int_T^\infty e_{\frac{2P}{1-\mu P}}(s,T)\frac{ww^\sigma}{1-\mu P}\Delta s \\ &\quad -\int_T^\infty e_{\frac{2P}{1-\mu P}}(s,T)\frac{P^2}{1-\mu P}\Delta s. \end{aligned} \tag{4.3}$$

Since the first improper integral in (4.3) converges we get

$$\begin{aligned} \infty &> \int_T^\infty e_{\frac{2P}{1-\mu P}}(s,T)\frac{ww^\sigma}{1-\mu P}\Delta s \\ &= \int_T^\infty \left(e_{\frac{2P}{1-\mu P}}w\right)\left(e^\sigma_{\frac{2P}{1-\mu P}}w^\sigma\right)e^\sigma_{\frac{-2P}{1+\mu P}}\frac{1}{1-\mu P}\Delta s \\ &= \int_T^\infty \left(e_{\frac{2P}{1-\mu P}}w\right)\left(e^\sigma_{\frac{2P}{1-\mu P}}w^\sigma\right)e_{\frac{-2P}{1+\mu P}}\frac{1}{1+\mu P}\Delta s \\ &\geq A^2\int_T^\infty e_{\frac{-2P}{1+\mu P}}\frac{1}{1+\mu P}\Delta s. \end{aligned}$$

Since $A>0$, we obtain

$$\int_T^\infty e_{\frac{-2P}{1+\mu P}}(t,T)\frac{1}{1+\mu(t)P(t)}\Delta t<\infty.$$

But then, since $\lim_{t\to\infty}[1+\mu(t)P(t)]=1$, we get that

$$\int_T^\infty e_{\frac{-2P}{1+\mu P}}(t,T)\Delta t<\infty. \tag{4.4}$$

Since the second improper integral in (4.3) converges, we find that (using $\lim_{t\to\infty}[1-\mu(t)P(t)]=1$)

$$\int_T^\infty P^2(t)e_{\frac{2P}{1-\mu P}}(t,T)\Delta t<\infty. \tag{4.5}$$

Using the Cauchy–Schwarz inequality we have that

$$\begin{aligned}\int_T^\infty |P(t)|\Delta s &= \int_T^\infty |P(t)|\left(e_{\frac{2P}{1-\mu P}}(t,T)\right)^{\frac12}\left(e_{\frac{-2P}{1+\mu P}}(t,T)\right)^{\frac12}\Delta t\\ &\le \left\{\int_T^\infty P^2(t)e_{\frac{2P}{1-\mu P}}(t,T)\Delta t\int_T^\infty e_{\frac{-2P}{1+\mu P}}(t,T)\Delta t\right\}^{\frac12}\\ &< \infty,\end{aligned} \tag{4.6}$$

where we used (4.4) and (4.5).

By the definition of $e_P(t,T)$, ([**5**], Page 57), we obtain

$$e_P(t,T)=\exp\left(\int_T^t \xi_{\mu(\tau)}(P(\tau))\Delta\tau\right), \tag{4.7}$$

where

$$\xi_h(P(\tau))=\begin{cases}\frac1h Log(1+hP(\tau)), & h>0\\ 0, & h=0.\end{cases} \tag{4.8}$$

and where Log is the principal logarithm function.

Note that $\lim_{x\to0}\frac{|\ln(1+x)|}{|x|}=1$, so $|\ln(1+x)|\le 2|x|$, for small x. So for $\mu(\tau)>0$, we have

$$|\xi_\mu(\tau)(P(\tau))|=\frac{1}{\mu(\tau)}|\ln(1+\mu(\tau)P(\tau))|\le 2|P(\tau)|,\text{ for large }\tau. \tag{4.9}$$

For $\mu(\tau)=0$, we have

$$|\xi_\mu(\tau)(P(\tau))|=|P(\tau)|\le 2|P(\tau)|.$$

So in any case, that is, for any $\mu(\tau)\ge0$, we have $|\xi_\mu(\tau)(P(\tau))|\le 2|P(\tau)|$. Therefore by (4.7), we get that

$$e_P(t,T)\le\exp\left(\int_T^t 2|P(\tau)|\Delta\tau\right)\le\int_T^\infty 2|P(\tau)|\Delta\tau=:M.$$

So

$$0<e^{-M}\le e_P(t,T)\le e^M.$$

Similarly we have $e_{-P}(t,T)$ is bounded. So we get that there exist constants $c_1>0, c_2>0$ such that

$$c_1\le\frac{e_{(-P)}(t,T)}{e_P(t,T)}=e_{\frac{-2P}{1+\mu P}}(t,T)\le c_2. \tag{4.10}$$

From (4.4), (4.10), we obtain that

$$\int_T^\infty e_{\frac{-4P}{1+\mu P}}(t,T)\Delta t<\infty. \tag{4.11}$$

But then from Theorem 3.1, it follows that equation (1.2) is oscillatory, which is a contradiction to our assumption. Hence, $A = 0$. Now, replacing T in (4.3) by t, we see that (4.1) follows from (4.3). This completes the proof. □

We may now establish our main oscillation result.

THEOREM 4.2. *Assume that*

$$\int_{t_0}^{\infty} P(t)\Delta t \quad \textit{and} \quad \int_{t_0}^{\infty} P^2(t)\Delta t$$

are both convergent. Suppose that $\mu(t)$ is bounded and that $\mathbb{T}$ satisfies condition (D). Let

$$\bar{P}(t) := \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s \quad \textit{and} \quad S(t) := \frac{2P(t)+4\bar{P}(t)}{1-\mu(t)P(t)}.$$

If for sufficiently large T

$$\int_T^{\infty} e_{\ominus S}(t,T)\Delta t < \infty, \tag{4.12}$$

then (1.2) *is oscillatory.*

REMARK 4.3. For $\mathbb{T} = \mathbb{R}$, the assumptions of Theorem A stated earlier in Section 1 imply that $\int_{t_0}^{\infty} P^2(t)\Delta t$ is convergent. Therefore, Theorem 3.1 may be considered as an extension of Theorem A.

PROOF. Assume (1.2) is nonoscillatory, let $x(t) > 0$ be a solution of (1.2), and let $z(t) = \frac{x^{\Delta}(t)}{x(t)}$ be the corresponding solution of the Riccati equation

$$z^{\Delta} + p(t) + \frac{z^2}{1+\mu(t)z} = 0.$$

on $[T,\infty)$. By Lemma 2.1, integrating the Riccati equation from t to ∞, we get that

$$z(t) = \int_t^{\infty} p(s)\Delta s + \int_t^{\infty} R(s)\Delta s,$$

where $R(s) = \frac{z^2(s)}{1+\mu(s)z(s)}$. If we define $w(t) := \int_t^{\infty} R(s)\Delta s > 0$, then we have $z(t) = P(t) + w(t)$. From Theorem 4.1, we get that $w(t)$ solves the integral equation

$$w(t) = \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\,\frac{P^2(s)+w(s)w(\sigma(s))}{1-\mu(s)P(s)}\Delta s. \tag{4.13}$$

for $t \geq T$. Letting

$$v(t) := \int_t^{\infty} e_{\frac{2P}{1-\mu P}}(s,t)\,\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\Delta s, \tag{4.14}$$

we have from (4.13) that

$$w(t) = \bar{P}(t) + v(t).$$

Using several properties of the generalized exponential function [**5**, Theorem 2.36] we see that

$$\begin{aligned} e_{\frac{2P}{1-\mu P}}(s,t) &= e_{\frac{2P}{1-\mu P}}(s,T)\, e_{\frac{2P}{1-\mu P}}(T,t) \\ &= e_{\frac{2P}{1-\mu P}}(s,T)\, e_{\frac{-2P}{1+\mu P}}(t,T) \\ &= \frac{e_P(s,T)}{e_{-P}(s,T)}\, \frac{e_{-P}(t,T)}{e_P(t,T)}. \end{aligned}$$

Hence, from (4.14), we have

$$v(t) = \frac{e_{-P}(t,T)}{e_P(t,T)} \int_t^\infty \frac{e_P(s,T)}{e_{-P}(s,T)}\, \frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)} \Delta s.$$

Then using the product rule and simplifying we get that

$$v^\Delta(t) = \frac{-2P(t)v(\sigma(t))}{1-\mu(t)P(t)} - \frac{w(t)w(\sigma(t))}{1-\mu(t)P(t)}.$$

Note that $w^\Delta(t) = -R(t) \leq 0$, so $w(t) \geq w(\sigma(t))$ and consequently

$$\begin{aligned} v^\Delta(t) &\leq \frac{-2P(t)v(\sigma(t))}{1-\mu(t)P(t)} - \frac{w^2(\sigma(t))}{1-\mu(t)P(t)} \\ &= \frac{-2P(t)v(\sigma(t))}{1-\mu(t)P(t)} - \frac{\left(\bar{P}^\sigma(t) + v^\sigma(t)\right)^2}{1-\mu(t)P(t)}. \end{aligned}$$

Using the inequality

$$\bar{P}^2(\sigma(t)) + v^2(\sigma(t)) \geq 2\bar{P}(\sigma(t))v(\sigma(t))$$

we get

$$v^\Delta(t) \leq S(t)v(\sigma(t)), \tag{4.15}$$

where

$$S(t) := \frac{2P(t) + 4\bar{P}(t)}{1-\mu(t)P(t)}.$$

Multiplying (4.15) by the integrating factor $e_S(t,T)$ and using the product rule we obtain

$$(v(t)e_S(t,T))^\Delta \leq 0.$$

Integrating from T to t, and solving for $v(t)$ we get that

$$v(t) \leq v(T)e_{\ominus S}(t,T).$$

Using (4.12), it follows that $\int_{t_0}^\infty v(t)\Delta t$ converges.

For $\mu(\tau) > 0$, by (4.8) and Taylor's formula, we get that

$$\begin{aligned}\xi_{\mu(\tau)}(P(\tau)) &= \frac{1}{\mu(\tau)}\ln(1+\mu(\tau)P(\tau)) \\ &= P(\tau) - \frac{\mu(\tau)P^2(\tau)}{2} + \mu(\tau)o(P^2(\tau)).\end{aligned}$$

This same formula holds when $\mu(\tau) = 0$, since we have in this case that $\xi_{\mu(\tau)}(P(\tau)) = P(\tau)$.

So in any case, we have for all $\mu(\tau)$ the formula

$$\xi_{\mu(\tau)}(P(\tau)) = P(\tau) - \frac{\mu(\tau)P^2(\tau)}{2} + \mu(\tau)o(P^2(\tau)).$$

Since $\int_{t_0}^{\infty} P(t)\Delta t$ and $\int_{t_0}^{\infty} P^2(t)\Delta t$ are both convergent, we get that

$$\int_T^{\infty} \xi_{\mu(\tau)}(P(\tau))\Delta\tau$$

is convergent. Similarly we have $\int_T^{\infty} \xi_{\mu(\tau)}(-P(\tau))\Delta\tau$ is also convergent. Therefore by the definitions of $e_{-P}(t,T)$ and $e_P(t,T)$, we get that there exist constants $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \le \frac{e_{-P}(t,T)}{e_P(t,T)} = e_{\frac{-2P}{1+\mu P}}(t,T) \le c_2. \tag{4.16}$$

We now choose T sufficiently large so that for $t \ge 2T$ we have (using (4.14)) that

$$\begin{aligned}\int_T^t v(\tau)\Delta\tau &= \int_T^t \int_\tau^{\infty} e_{\frac{2P}{1-\mu P}}(s,\tau)\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\Delta s\Delta\tau \\ &= \int_T^t e_{\frac{-2P}{1+\mu P}}(\tau,T)\int_\tau^{\infty} e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\Delta s\Delta\tau \\ &\ge \int_T^t \left[\int_T^s e_{\frac{-2P}{1+\mu P}}(\tau,T)\Delta\tau\right]\left[e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\right]\Delta s \\ &\ge \int_T^t c_1(s-T)\left[e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\right]\Delta s \\ &\ge \frac{c_1}{2}\int_{2T}^t s\, e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\Delta s. \qquad (4.17)\end{aligned}$$

The Cauchy–Schwarz inequality shows that

$$\begin{aligned}&\left[\int_{2T}^t \sqrt{w(s)w(\sigma(s))}\Delta s\right]^2 \qquad (4.18)\\ &\le \int_{2T}^t s\, e_{\frac{2P}{1-\mu P}}(s,T)\frac{w(s)w(\sigma(s))}{1-\mu(s)P(s)}\Delta s \int_{2T}^t \frac{1-\mu(s)P(s)}{s} e_{\frac{-2P}{1+\mu P}}(s,T)\Delta s.\end{aligned}$$

Now since $\mu(t)$ is bounded, we have that $\lim_{t\to\infty}[1-\mu(t)P(t)] = 1$ and $w^{\Delta}(t) = -R(t) \le 0$. Hence, using (4.16)–(4.18) and Lemma 2.3, we obtain

$$\int_{2T}^{t} w(\sigma(s))\Delta s \le \int_{2T}^{t} \sqrt{w(s)w(\sigma(s))}\Delta s \le c_3 \left(\ln \frac{t}{2T}\right)^{\frac{1}{2}}. \tag{4.19}$$

Since $\mu(t)$ is bounded and $\lim_{t\to\infty} z(t) = 0$, there exists $c_4 > 0$, such that $z^2(s) \le c_4 R(s)$. From Lemma 2.3 and (4.19), we get

$$\begin{aligned}
\left(\ln \frac{x(t)}{x(2T)}\right)^2 &\le \left(\int_{2T}^{t} z(s)\Delta s\right)^2 \\
&\le \left(\int_{2T}^{t} sz^2(s)\Delta s\right)\int_{2T}^{t} \frac{1}{s}\Delta s \\
&\le c_4\left(\int_{2T}^{t} sR(s)\Delta s\right)\int_{2T}^{t} \frac{1}{s}\Delta s \\
&= c_4\left(-\int_{2T}^{t} sw^{\Delta}(s)\Delta s\right)\int_{2T}^{t} \frac{1}{s}\Delta s \\
&= c_4\left(2Tw(2T) + \int_{2T}^{t} w(\sigma(s))\Delta s\right)\int_{2T}^{t} \frac{1}{s}\Delta s \\
&\le c_5(\ln \frac{t}{2T})^{\frac{3}{2}}.
\end{aligned}$$

So

$$\begin{aligned}
x(t) &\le x(2T)\exp\left\{c_6\left(\ln\frac{t}{2T}\right)^{\frac{3}{4}}\right\} \\
&\le c_7 \exp\left(\frac{1}{2}\ln t\right) = c_7\sqrt{t}, \quad t \ge 2T.
\end{aligned}$$

Hence

$$\frac{1}{x(t)} \ge \frac{1}{c_7 t^{\frac{1}{2}}} \quad \text{and} \quad \frac{1}{x(\sigma(t))} \ge \frac{1}{c_7(\sigma(t))^{\frac{1}{2}}}.$$

Since $\mu(t)$ is bounded, there exists $M_1 > 0$ such that

$$\int^{\infty} \frac{1}{x(t)x(\sigma(t))}\Delta t \ge \frac{1}{c_7^2}\int^{\infty} \frac{1}{t^{\frac{1}{2}}(t+M_1)^{\frac{1}{2}}}\Delta t = \infty,$$

which contradicts the existence of dominant solutions (see Lemma 2.4). This completes the proof. □

When $\mathbb{T} = \mathbb{N}$ in Theorem 4.2 after several routine calculations we get the following result.

COROLLARY 4.4. *Assume that* $\sum_{i=1}^{\infty} p_i$ *is convergent,* $P_n := \sum_{i=n}^{\infty} p_i$, $\sum_{i=1}^{\infty} P_i$ *and* $\sum_{i=1}^{\infty} P_i^2$ *are convergent. Let* $N \ge 1$ *be so large that* $|P_n| < 1$, *for* $n \ge N$. *Let us set*

$$q_n := \prod_{j=N}^{n-1} \frac{1-P_j}{1+P_j}, \quad g(j;n) := \frac{q_n}{q_{j+1}(1+P_j)},$$

$$\bar{P}_n := \sum_{j=n}^{\infty} g(j;n) P_j^2, \quad \textit{for } j \geq n \geq N.$$

$$S_n := \frac{2P_n + 4\bar{P}_n}{1 - P_n}$$

If

$$\sum_{j=N+1}^{\infty} \prod_{i=N}^{j-1} (1 + S_i)^{-1} < +\infty,$$

then $x^{\Delta\Delta}(n) + p_n x(n+1) = 0$ *is oscillatory.*

5. Example

In this example we use Corollary 4.4 to show that

$$x^{\Delta\Delta}(n) + \frac{b(-1)^n}{n} x(n+1) = 0$$

is oscillatory when $|b| > 1$. Let

$$p_n := \frac{b(-1)^n}{n}, \; n \in \mathbb{T} = \mathbb{N}, \; b \neq 0 \in \mathbb{R}.$$

Let $P_n := \sum_{j=n}^{\infty} p_j$. We have

$$\begin{aligned} P_{2k} &= \sum_{j=2k}^{\infty} p_j \\ &= b\left(\frac{1}{2k} - \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{2k+3} + \cdots\right) \\ &= b\left(\frac{1}{2k(2k+1)} + \frac{1}{(2k+2)(2k+3)} + \cdots\right). \end{aligned}$$

(5.1)

It follows that

$$\frac{|b|}{4} \sum_{j=k}^{\infty} \frac{1}{j(j+1)} \leq |P_{2k}| \leq \frac{|b|}{4} \sum_{j=k}^{\infty} \frac{1}{j^2}$$

and hence we have

$$P_{2k} \sim \frac{b}{4k} = \frac{b}{2} \frac{1}{2k}.$$

Similarly we have

$$P_{2k+1} \sim -\frac{b}{2} \frac{1}{2k+1},$$

so

$$P_n \sim (-1)^n \frac{b}{2} \frac{1}{n}. \tag{5.2}$$

Therefore $\sum_{k=n}^{\infty} P_k$ converges.

Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, we have that $\sum_{k=n}^{\infty} P_k^2$ converges.

Using $\ln(1+x) = x - \frac{1}{2}x^2 + \mathrm{o}(x^2)$, we have

$$\ln\left(1 - \frac{2P_j}{1+P_j}\right) = -\frac{2P_j}{1+P_j} - \frac{1}{2}\left(\frac{2P_j}{1+P_j}\right)^2 + \mathrm{o}\left(\left(\frac{2P_j}{1+P_j}\right)^2\right).$$

Also, we have

$$\frac{P_j}{1+P_j} = P_j(1 - P_j + \mathrm{O}(P_j^2)).$$

So

$$\sum_{j=N}^{\infty} \ln\left(\frac{1-P_j}{1+P_j}\right) = \sum_{j=N}^{\infty} \ln\left(1 - \frac{2P_j}{1+P_j}\right)$$

is convergent. Thus,

$$q_n = \prod_{N}^{n-1} \frac{1-P_j}{1+P_j} = \exp\left(\sum_{j=N}^{n-1} \ln\left(1 - \frac{2P_j}{1+P_j}\right)\right) > 1 - \epsilon, \text{for large } N.$$

We also have

$$g(j,n) = \frac{q_n}{q_{j+1}(1+P_j)} \geq (1-\epsilon_1), \quad j \geq n \geq N,$$

where we used $P_j \to 0$, $q_j \to 1$. By (5.2), we get (using $P_j^2 \geq (1-\epsilon_2)\frac{b^2}{4}(\frac{1}{j})^2$) that

$$\begin{aligned} \bar{P}_n &= \sum_{j=n}^{\infty} g(j,n) P_j^2 \geq (1-\epsilon_1) \sum_{j=n}^{\infty} P_j^2 \\ &\geq (1-\epsilon_1)(1-\epsilon_2)(\frac{b}{2})^2(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots) \\ &\geq (1-\epsilon_1)(1-\epsilon_2)(\frac{b}{2})^2(1-\epsilon_3)\frac{1}{n} \\ &= (1-\epsilon')(\frac{b}{2})^2\frac{1}{n}. \end{aligned} \tag{5.3}$$

By Taylor's formula, using (5.2) and (5.3), we have that for $n > N$,

$$\begin{aligned}
& \ln \prod_{i=N}^{n-1} (1+S_i)^{-1} \\
= \;& \ln \prod_{i=N}^{n-1} \left(1 + \frac{2P(i) + 4\bar{P}(i)}{1-P(i)}\right)^{-1} \\
= \;& \sum_{i=N}^{n-1} \ln \frac{1-P(i)}{1+P(i)+4\bar{P}(i)} \\
= \;& \sum_{i=N}^{n-1} [\ln(1-P(i)) - \ln(1+P(i)+4\bar{P}(i)) \\
= \;& \sum_{i=N}^{n-1} [-P(i) - \frac{1}{2}P^2(i) + o(P^2(i))] \\
- \;& \sum_{i=N}^{n-1} [-(P(i)+4\bar{P}(i)) - \frac{1}{2}(P(i)+4\bar{P}(i))^2 + o((P^2(i)+\bar{P}(i))^2)] \\
= \;& \sum_{i=N}^{n-1} [-4\bar{P}(i) + O(\frac{1}{i^2})] \\
\leq \;& \sum_{i=N}^{n-1} [-\frac{(1-\epsilon')b^2}{i} + O(\frac{1}{i^2})].
\end{aligned}$$

So using the Euler Formula [**13**, page 205]

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right)$$

we have

$$\begin{aligned}
\prod_{i=N}^{n-1} (1+S_i)^{-1} &\leq C_1 \exp\left[\sum_{i=N}^{n-1} \frac{-(1-\epsilon')b^2}{i} \right] \\
&\leq C_2 \exp\left[-(1-\epsilon')b^2(\gamma + \ln n - \sum_{i=1}^{N-1} \frac{1}{i}) \right]
\end{aligned}$$

where γ is the Euler constant. Therefore

$$\prod_{i=N}^{n-1} (1+S_i)^{-1} \leq C_3 n^{-(1-\epsilon')b^2}.$$

So when $|b| > 1$,

$$\sum_{j=N+1}^{\infty} \prod_{i=N}^{j-1} (1+S_i)^{-1} < \infty,$$

and so by Corollary 4.4, $x^{\Delta\Delta}(n) + \frac{b(-1)^n}{n}x(n+1) = 0$ is oscillatory.

Remark 5.1. In [**1**], it was shown that in the above example, the constant 1 is a critical value, i.e. when $|b| \leq 1$, $x^{\Delta\Delta}(n) + \frac{b(-1)^n}{n}x(n+1) = 0$ is nonoscillatory.

Remark 5.2. It should be pointed out that in the proof of Theorem A given in Wong [**3**, page 202, lines 12-13], $\int_0^T u(s)ds \leq c_4 T^{\frac{1}{2}}$ should read $\int_0^T u(s)ds \leq c_4$. The reasoning will also be more transparent if $\frac{3}{4}$ in statement (37) on the same page is replaced by $\frac{1}{2}$.

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