INEQUALITIES ON TIME SCALES: A SURVEY

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Abstract. The study of dynamic equations on time scales, which goes back to its founder Stefan Hilger (1988), is an area of mathematics which is currently receiving considerable attention. Although the basic aim of this is to unify the study of differential and difference equations, it also extends these classical cases to "in between". In this paper we present time scales versions of the inequalities: Hölder, Cauchy-Schwarz, Minkowski, Jensen, Gronwall, Bernoulli, Bihari, Opial, Wirtinger, and Lyapunov.

1. Unifying Continuous and Discrete Analysis

In 1988, Stefan Hilger [13] introduced the calculus on *time scales* which unifies continuous and discrete analysis. A time scale is a closed subset of the real numbers. We denote a time scale by the symbol $\mathbb{T}$. For functions y defined on $\mathbb{T}$, we introduce a so-called delta derivative y^Δ. This delta derivative is equal to y' (the usual derivative) if $\mathbb{T} = \mathbb{R}$ is the set of all real numbers, and it is equal to Δy (the usual forward difference) if $\mathbb{T} = \mathbb{Z}$ is the set of all integers. Then we study *dynamic equations*

$$f(t, y, y^\Delta, y^{\Delta^2}, \ldots, y^{\Delta^n}) = 0,$$

which may involve higher order derivatives as indicated. Along with such dynamic equations we consider initial values and boundary conditions. We remark that these

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dynamic equations are differential equations when $\mathbb{T} = \mathbb{R}$ and difference equations when $\mathbb{T} = \mathbb{Z}$. Other kinds of equations are covered by them as well, such as *q-difference equations*

$$\mathbb{T} = q^{\mathbb{Z}} := \{q^k |\ k \in \mathbb{Z}\} \cup \{0\} \quad \text{for some} \quad q > 1$$

and difference equations with constant step size

$$\mathbb{T} = h\mathbb{Z} := \{hk |\ k \in \mathbb{Z}\} \quad \text{for some} \quad h > 0.$$

Particularly useful for the discretization purpose are time scales of the form

$$\mathbb{T} = \{t_k |\ k \in \mathbb{Z}\} \quad \text{where} \quad t_k \in \mathbb{R},\ t_k < t_{k+1} \text{ for all } k \in \mathbb{Z}.$$

This survey paper is organized as follows: In Section 2 we introduce the basic concepts of the time scales calculus. Section 3 contains the Cauchy-Schwarz, Hölder and Minkowski inequalities, which can be proved by following the methods similar to those described in [12], see also the paper of Bohner and Lutz [8]. In Section 4 we derive a time scale version of Jensen's inequality. The obtained inequality reduces to the classical Jensen inequality in case $\mathbb{T} = \mathbb{R}$, and becomes the arithmetic-mean geometric-mean inequality in case $\mathbb{T} = \mathbb{Z}$. Next, in Section 5 we state the Gronwall inequality, which is essentially due to Hilger [13], and discuss its several interesting special cases. In particular, we shall show that this inequality reduces to the well-known Bernoulli inequality in case $\mathbb{T} = \mathbb{Z}$. A Bihari type inequality by Özgün, Kaymakçalan, and Zafer [16] is mentioned as well. Following the lead of Bohner and Kaymakcalan [7] and Hilscher [15], in Section 6 we obtain Opial and Wirtinger type inequalities. Finally, in Section 5 we present Lyapunov type inequalities and give some of their applications. This supplements the recent work of Bohner, Clark and Ridenhour [6].

2. The Time Scales Calculus

A time scale is a closed subset of the reals, and we usually denote it by the symbol $\mathbb{T}$. The two most popular examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} |\ s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} |\ s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ is called *right-scattered*, *right-dense*, *left-scattered*, *left-dense*, if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, $\rho(t) = t$ holds, respectively. The set $\mathbb{T}^\kappa$ is defined to be $\mathbb{T}$ if $\mathbb{T}$ does not have a left-scattered maximum; otherwise it is $\mathbb{T}$ without this left-scattered maximum. The *graininess* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Hence the graininess function is constant 0 if $\mathbb{T} = \mathbb{R}$ while it is constant 1 for $\mathbb{T} = \mathbb{Z}$. However, a time scale $\mathbb{T}$ could have nonconstant graininess. Now, let f be a function defined on $\mathbb{T}$. We say that f is *delta differentiable* (or simply: *differentiable*) at $t \in \mathbb{T}^\kappa$ provided there exists an α such that for all $\varepsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in \mathcal{N}.$$

In this case we denote the α by $f^\Delta(t)$, and if f is differentiable for every $t \in \mathbb{T}^\kappa$, then f is said to be *differentiable* on $\mathbb{T}$ and f^Δ is a new function defined on $\mathbb{T}^\kappa$. If f is differentiable at $t \in \mathbb{T}^\kappa$, then it is easy to see that

$$f^\Delta(t) = \begin{cases} \lim_{s \to t, s \in \mathbb{T}} \frac{f(t) - f(s)}{t - s} & \text{if} \quad \mu(t) = 0 \\ \frac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if} \quad \mu(t) > 0. \end{cases}$$

Other useful formulas are as follows:

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t) \tag{2.1}$$

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t) \tag{2.2}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g(\sigma(t))}. \tag{2.3}$$

A function f defined on $\mathbb{T}$ is *rd-continuous*, if it is continuous at every right-dense point and if the left-sided limit exists in every left-dense point. The importance of rd-continuous functions is revealed by the following existence result by Hilger [13]: *Every rd-continuous function possesses an antiderivative.* Here, F is called an antiderivative of a function f defined on $\mathbb{T}$ if $F^{\Delta} = f$ holds on $\mathbb{T}^{\kappa}$. In this case we define an integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \quad \text{where} \quad s, t \in \mathbb{T}.$$

For further basic results about the time scales calculus we refer the reader to [3, 1, 13, 5].

3. Hölder's Inequality

The following version of Hölder's inequality on time scales appears in [8, Lemma 2.2 (iv)], and it's proof is similar to that of the classical inequality as given e.g. in [12, Theorem 188].

Theorem 3.1 (Hölder's Inequality)**.** *Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \le \left\{\int_a^b |f(t)|^p \Delta t\right\}^{\frac{1}{p}} \left\{\int_a^b |g(t)|^q \Delta t\right\}^{\frac{1}{q}},$$

where $p > 1$ and $q = p/(p-1)$.

Proof. For nonnegative real numbers α and β, the basic inequality

$$\alpha^{1/p}\beta^{1/q} \leq \frac{\alpha}{p} + \frac{\beta}{q} \tag{3.1}$$

holds. Now suppose, without loss of generality, that

$$\left\{\int_a^b |f(t)|^p \Delta t\right\}\left\{\int_a^b |g(t)|^q \Delta t\right\} \neq 0.$$

Apply (3.1) to

$$\alpha(t) = \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \Delta\tau} \quad \text{and} \quad \beta(t) = \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \Delta\tau}$$

and integrate the obtained inequality between a and b (this is possible since all occuring functions are rd-continuous) to find

$$\begin{aligned}
&\int_a^b \frac{|f(t)|}{\left\{\int_a^b |f(\tau)|^p \Delta\tau\right\}^{1/p}} \frac{|g(t)|}{\left\{\int_a^b |g(\tau)|^q \Delta\tau\right\}^{1/q}} \Delta t = \int_a^b \alpha^{1/p}(t)\beta^{1/q}(t)\Delta t \\
\leq\ & \int_a^b \left\{\frac{\alpha}{p} + \frac{\beta}{q}\right\} \\
=\ & \int_a^b \left\{\frac{1}{p}\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \Delta\tau} + \frac{1}{q}\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \Delta\tau}\right\}\Delta t \\
=\ & \frac{1}{p}\int_a^b \left\{\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p \Delta\tau}\right\}\Delta t + \frac{1}{q}\int_a^b \left\{\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q \Delta\tau}\right\}\Delta t \\
=\ & \frac{1}{p} + \frac{1}{q} \\
=\ & 1.
\end{aligned}$$

This directly yields Hölder's inequality. $\square$

The special case $p = q = 2$ reduces to the Cauchy-Schwarz inequality.

Theorem 3.2 (Cauchy-Schwarz Inequality). *Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(t)g(t)|\Delta t \leq \sqrt{\left\{\int_a^b |f(t)|^2 \Delta t\right\}\left\{\int_a^b |g(t)|^2 \Delta t\right\}}.$$

Next, we can use Hölder's inequality to deduce Minkowski's inequality.

Theorem 3.3 (Minkowski's Inequality). *Let $a, b \in \mathbb{T}$ and $p > 1$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$ we have*

$$\left\{\int_a^b |(f+g)(t)|^p \Delta t\right\}^{1/p} \leq \left\{\int_a^b |f(t)|^p \Delta t\right\}^{1/p} + \left\{\int_a^b |g(t)|^p \Delta t\right\}^{1/p}.$$

Proof. We apply Hölder's inequality, Theorem 3.1, with $q = p/(p-1)$ to obtain

$$\begin{aligned}
\int_a^b |(f+g)(t)|^p \Delta t &= \int_a^b |(f+g)|^{p-1} |(f+g)(t)|\Delta t \\
&\leq \int_a^b |f(t)||(f+g)(t)|^{p-1}\Delta t + \int_a^b |g(t)||f+g|^{p-1}(t)\Delta t \\
&\leq \left\{\int_a^b |f(t)|^p \Delta t\right\}^{1/p} \left\{\int_a^b |(f+g)(t)|^{(p-1)q}\Delta t\right\}^{1/q} \\
&\qquad + \left\{\int_a^b |g(t)|^p \Delta t\right\}^{1/p} \left\{\int_a^b |(f+g)(t)|^{(p-1)q}\Delta t\right\}^{1/q} \\
&= \left[\left\{\int_a^b |f(t)|^p \Delta t\right\}^{1/p} + \left\{\int_a^b |g(t)|^p \Delta t\right\}^{1/p}\right]\left[\int_a^b |(f+g)(t)|^p \Delta t\right]^{1/q}.
\end{aligned}$$

We divide both sides of the obtained inequality by $\left[\int_a^b |(f+g)(t)|^p \Delta t\right]^{1/q}$ to arrive at Minkowski's inequality. □

4. Jensen's Inequality

The proof of Jensen's inequality on time scales follows closely to the proof of the classical Jensen's inequality (see for example [11, Problem 3.42]). If $\mathbb{T} = \mathbb{R}$, then our version is the same as the classical Jensen inequality. However, if $\mathbb{T} = \mathbb{Z}$, then it reduces to the well-known arithmetic-mean geometric-mean inequality.

Theorem 4.1 (Jensen's Inequality)**.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose $g : [a, b] \to (c, d)$ is rd-continuous and $F : (c, d) \to \mathbb{R}$ is convex. Then*

$$F\left(\frac{\int_a^b g(t)\Delta t}{b-a}\right) \leq \frac{\int_a^b F(g(t))\Delta t}{b-a}.$$

Proof. Let $x_0 \in (c, d)$. Then (e.g., by [11, p. 109]) there exists $\beta \in \mathbb{R}$ such that

$$F(x) - F(x_0) \geq \beta(x - x_0) \quad \text{for all} \quad x \in (c, d). \tag{4.1}$$

Since g is rd-continuous,

$$x_0 = \frac{\int_a^b g(\tau)\Delta\tau}{b-a}$$

is well-defined. $F \circ g$ is also rd-continuous, and hence we may apply (4.1) with $x = g(t)$ and integrate from a to b to obtain

$$\begin{aligned}
\int_a^b F(g(t))\Delta t - (b-a)F\left(\frac{\int_a^b g(\tau)\Delta\tau}{b-a}\right) &= \int_a^b F(g(t))\Delta t - (b-a)F(x_0) \\
&= \int_a^b [F(g(t)) - F(x_0)] \\
&\geq \beta \int_a^b [g(t) - x_0]\,\Delta t \\
&= \beta\left[\int_a^b g(t)\Delta t - x_0(b-a)\right] \\
&= 0.
\end{aligned}$$

This directly yields Jensen's inequality. $\square$

Example 4.1. Let $\mathbb{T} = \mathbb{R}$. Obviously, $F = -\log$ is convex and continuous on $(0, \infty)$, so we may apply Theorem 4.1 with $a = 0$ and $b = 1$ to obtain

$$\log \int_0^1 g(t)\mathrm{d}t \geq \int_0^1 \log g(t)\mathrm{d}t$$

and hence

$$\int_0^1 g(t)\mathrm{d}t \geq \exp\left[\int_0^1 \log g(t)\mathrm{d}t\right]$$

whenever $g : [0,1] \to (0, \infty)$ is continuous.

Example 4.2. Let $\mathbb{T} = \mathbb{Z}$ and $N \in \mathbb{N}$. Again we apply Jensen's inequality, Theorem 4.1, with $a = 1$, $b = N+1$, and $g : \{1, 2, \ldots, N+1\} \to (0, \infty)$ to find

$$\begin{aligned}
\log\left\{\frac{1}{N}\sum_{t=1}^{N} g(t)\right\} &= \log\left\{\frac{1}{n}\int_1^{N+1} g(t)\Delta t\right\} \\
&\geq \frac{1}{N}\int_1^{N+1} \log g(t)\Delta t \\
&= \frac{1}{N}\sum_{t=1}^{N} \log g(t) \\
&= \log\left\{\prod_{t=1}^{N} g(t)\right\}^{1/N}.
\end{aligned}$$

and hence

$$\frac{1}{N}\sum_{t=1}^{N} g(t) \geq \left\{\prod_{t=1}^{N} g(t)\right\}^{1/N}.$$

This is the well-known *arithmetic-mean geometric-mean inequality.*

Example 4.3. Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $N \in \mathbb{N}$. We apply Theorem 4.1 with $a = 1$, $b = 2^N$, and $g : \{2^k |\ 0 \le k \le N\} \to (0, \infty)$ to find

$$\begin{aligned}
\log\left\{\frac{\sum_{k=0}^{N-1} 2^k g(2^k)}{2^N - 1}\right\} &= \log\left\{\frac{\int_1^{2^N} g(t)\Delta t}{2^N - 1}\right\} \\
&\ge \frac{\int_1^{2^N} \log g(t)\Delta t}{2^N - 1} \\
&= \frac{\sum_{k=0}^{N-1} 2^k \log g(2^k)}{2^N - 1} \\
&= \frac{\sum_{k=0}^{N-1} \log(g(2^k))^{2^k}}{2^N - 1} \\
&= \frac{\log\left\{\prod_{k=0}^{N-1} (g(2^k))^{2^k}\right\}}{2^N - 1} \\
&= \log\left\{\prod_{k=0}^{N-1} (g(2^k))^{2^k}\right\}^{1/(2^N - 1)}
\end{aligned}$$

and therefore

$$\frac{\sum_{k=0}^{N-1} 2^k g(2^k)}{2^N - 1} \ge \left\{\prod_{k=0}^{N-1} (g(2^k))^{2^k}\right\}^{1/(2^N - 1)}.$$

5. Gronwall's Inequality

Definition 5.1. An rd-continuous function f is called *regressive* provided

$$1 + \mu(t) f(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

The set of all rd-continuous functions f that satisfy $1 + \mu(t) f(t) > 0$ for all $t \in \mathbb{T}$ will be denoted by $\mathcal{R}^+$.

The following existence theorem has been proved by Hilger, see [13].

Theorem 5.1. *Let $t_0 \in \mathbb{T}$. If p is rd-continuous and regressive, then*

$$y^\Delta = p(t) y, \quad y(t_0) = 1$$

has a unique solution.

We call the unique solution from Theorem 5.1 the *exponential function* and denote it by $\mathrm{e}_p(\cdot, t_0)$. In fact, there is an explicit formula for $\mathrm{e}_p(t,s)$, using the so-called cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{z} & \text{if} \quad h \neq 0 \\ z \quad \text{if} \quad h = 0. \end{cases}$$

The formula, see [14], reads

$$\mathrm{e}_p(t,s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\}.$$

We now proceed to give some fundamental properties of the exponenetial function. To do so it is necessary to introduce the following notation: For regressive $p, q : \mathbb{T} \to \mathbb{R}$ we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1+\mu p}, \quad p \ominus q := p \oplus (\ominus p).$$

Note that the set of all regressive and rd-continuous functions together with the addition $\oplus$ is an Abelian group.

Theorem 5.2. *Assume $p, q : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous, then the following hold:*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$*;*

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(vi) $e_p(t,s)e_q(t,s) = e_{p \oplus q}(t,s)$*;*

(vii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

Next we note the following result from [9].

Theorem 5.3. *If p and f are rd-continuous and p is regressive, then the unique solution of the initial value problem*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0$$

is given by

$$y(t) = y_0 \, \mathrm{e}_p(t,t_0) + \int_{t_0}^t \mathrm{e}_p(t,\sigma(\tau))f(\tau)\Delta\tau.$$

A comparison theorem follows.

Theorem 5.4. *Let y and f be rd-continuous and $p \in \mathcal{R}^+$. Then*

$$y^\Delta(t) \le p(t)y(t) + f(t) \quad \textit{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \le y(a)\, \mathrm{e}_p(t,a) + \int_a^t \mathrm{e}_p(t,\sigma(\tau))f(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}.$$

Proof. Note that $p \in \mathcal{R}^+$ implies $1 + \mu(t)p(t) > 0$ and hence $\mathrm{e}_p > 0$. Now

$$\begin{aligned}
[y\,\mathrm{e}_{\ominus p}(\cdot,a)]^\Delta(t) &= y^\Delta(t)\,\mathrm{e}_{\ominus p}(\sigma(t),a) + y(t)(\ominus p)(t)\,\mathrm{e}_{\ominus p}(t,a) \\
&= y^\Delta(t)[1+\mu(t)(\ominus p)(t)]\,\mathrm{e}_{\ominus p}(t,a) + y(t)(\ominus p)(t)\,\mathrm{e}_{\ominus p}(t,a) \\
&= \left[y^\Delta(t)\frac{1}{1+\mu(t)p(t)} - \frac{p(t)}{1+\mu(t)p(t)}y(t)\right]\mathrm{e}_{\ominus p}(t,a) \\
&= [y^\Delta(t) - p(t)y(t)]\frac{\mathrm{e}_{\ominus p}(t,a)}{1+\mu(t)p(t)} \\
&= [y^\Delta(t) - p(t)y(t)]\,\mathrm{e}_{\ominus p}(\sigma(t),a).
\end{aligned}$$

Therefore

$$\begin{aligned} y(t)\,\mathrm{e}_{\ominus p}(t,a)-y(a) &= \int_a^t [y^\Delta(\tau)-p(\tau)y(\tau)]\,\mathrm{e}_{\ominus p}(\sigma(\tau),a)\Delta\tau \\ &\le \int_a^t f(\tau)\,\mathrm{e}_{\ominus p}(\sigma(\tau),a)\Delta\tau \\ &= \int_a^t \mathrm{e}_p(a,\sigma(\tau))f(\tau)\Delta\tau \end{aligned}$$

and hence the assertion follows by applying Theorem 5.2. □

The above comparison Theorem 5.4 gives the following interesting results.

Theorem 5.5 (Bernoulli's Inequality)**.** *Let $\alpha \in \mathbb{R}$ with $\alpha \in \mathcal{R}^+$. Then*

$$\mathrm{e}_\alpha(t,s) \ge 1+\alpha(t-s) \quad \textit{for all} \quad t,s\in\mathbb{T}.$$

Proof. Since $\alpha \in \mathcal{R}^+$, we have $\mathrm{e}_\alpha(t,s)>0$ for all $t,s\in\mathbb{T}$. First, suppose $t\ge s$. Let $y(t)=\alpha(t-s)$. Then

$$\alpha y(t)+\alpha = \alpha^2(t-s)+\alpha \ge \alpha = y^\Delta(t).$$

Since $y(s)=0$, we have by Theorem 5.4

$$y(t) \le \int_s^t \mathrm{e}_\alpha(t,\sigma(\tau))\alpha\Delta\tau = -1+\mathrm{e}_\alpha(t,s)$$

so that $\mathrm{e}_\alpha(t,s)\ge 1+\alpha(t-s)$ follows. □

Theorem 5.6 (Gronwall's Inequality)**.** *Let y and f be rd-continuous and $p\in\mathcal{R}^+$, $p\ge 0$. Then*

$$y(t) \le f(t)+\int_a^t y(\tau)p(\tau)\Delta\tau \quad \textit{for all} \quad t\in\mathbb{T}$$

implies

$$y(t) \le f(t)+\int_a^t \mathrm{e}_p(t,\sigma(\tau))f(\tau)p(\tau)\Delta\tau \quad \textit{for all} \quad t\in\mathbb{T}.$$

Proof. Define

$$z(t) = \int_a^t y(\tau)p(\tau)\Delta\tau.$$

Then $z(a) = 0$ and

$$z^{\Delta}(t) = y(t)p(t) \le [f(t) + z(t)]p(t) = p(t)z(t) + p(t)f(t).$$

By Theorem 5.4,

$$z(t) \le \int_a^t \mathrm{e}_p(t, \sigma(\tau))f(\tau)p(\tau)\Delta\tau,$$

and hence the claim follows because of $y(t) \le f(t) + z(t)$. $\square$

If we take $\mathbb{T} = h\mathbb{Z}$ and $a = 0$ in Gronwall's inequality we get the following.

Example 5.1. If $\mathbb{T} = h\mathbb{Z}$ and y is a nonnegative function on $[0, \infty)$ and $b > 0$ is a constant such that

$$y(t) \le c(t) + b \sum_{k=0}^{\frac{t}{h}-1} y(kh)$$

for $t \in [0, \infty)$, then

$$y(t) \le c(t) + b \sum_{k=0}^{\frac{t}{h}-1} c(kh)(1 + bh)^{\frac{t-h(k+1)}{h}}.$$

If we let $h = 1$ in the above example we get the following.

Example 5.2. Assume $\{y_n\}_{n=0}^{\infty}$ is a sequence of nonnegative real numbers and $b > 0$ is a constant such that

$$y_n \le c_n + b \sum_{k=0}^{n-1} y(k)$$

for $n \in \mathbb{N}_0$, then it follows that

$$y_n \le c_n + b \sum_{k=0}^{n-1} c_k(1 + b)^{n-k-1}.$$

Corollary 5.1. *Let y be rd-continuous and $p \in \mathcal{R}^+$ with $p \geq 0$. Then*

$$y(t) \leq \int_a^t y(\tau)p(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \leq 0 \quad \textit{for all} \quad t \in \mathbb{T}.$$

Proof. This is Theorem 5.6 with $f(t) \equiv 0$. □

Corollary 5.2. *Let y be rd-continuous and $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma > 0$. Then*

$$y(t) \leq \alpha(t-a) + \beta + \gamma \int_a^t y(\tau)\Delta\tau \quad \textit{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \leq \left(\frac{\alpha}{\gamma} + \beta\right) \mathrm{e}_\gamma(t,a) - \frac{\alpha}{\gamma}.$$

Proof. In Theorem 5.6, let $f(t) = \alpha(t-a) + \beta$ and $p(t) \equiv \gamma$. Then

$$\begin{aligned} y(t) &\leq f(t) + \int_a^t \mathrm{e}_p(t,\sigma(\tau)) f(\tau) p(\tau)\Delta\tau \\ &= \alpha(t-a) + \beta + \gamma \int_a^t \mathrm{e}_\gamma(t,\sigma(\tau))[\alpha(\tau-a)+\beta]\Delta\tau. \end{aligned}$$

To proceed we observe that for

$$w(\tau) = -\frac{\mathrm{e}_\gamma(t,\tau)}{\gamma}$$

we have

$$\begin{aligned} w^\Delta(\tau) &= -\frac{1}{\gamma}(\ominus\gamma)(\tau)\, \mathrm{e}_{\ominus\gamma}(\tau,t) \\ &= \frac{1}{1+\mu(\tau)\gamma}\, \mathrm{e}_{\ominus\gamma}(\tau,t) \\ &= [1+\mu(\tau)(\ominus\gamma)(\tau)]\, \mathrm{e}_{\ominus\gamma}(\tau,t) \\ &= \mathrm{e}_{\ominus\gamma}(\sigma(\tau),t) \;=\; e_\gamma(t,\sigma(\tau)). \end{aligned}$$

Hence

$$
\begin{aligned}
\int_a^t \mathrm{e}_\gamma(t,\sigma(\tau))[\alpha(\tau-a)+\beta]\Delta\tau &= \int_a^t w^\Delta(\tau)[\alpha(\tau-a)+\beta]\Delta\tau \\
&= \int_a^t \left\{[w(\tau)(\alpha(\tau-a)+\beta)]^\Delta - \alpha w(\sigma(\tau))\right\}\Delta\tau \\
&= w(t)[\alpha(t-a)+\beta] - w(a)\beta - \alpha\int_a^t w(\sigma(\tau))\Delta\tau \\
&= -\frac{1}{\gamma}[\alpha(t-a)+\beta] + \frac{\beta}{\gamma}\mathrm{e}_\gamma(t,a) + \frac{\alpha}{\gamma}\int_a^t \mathrm{e}_\gamma(t,\sigma(\tau))\Delta\tau \\
&= -\frac{1}{\gamma}[\alpha(t-a)+\beta] + \frac{\beta}{\gamma}\mathrm{e}_\gamma(t,a) + \frac{\alpha}{\gamma}\int_a^t w^\Delta(\tau)\Delta\tau \\
&= -\frac{1}{\gamma}[\alpha(t-a)+\beta] + \frac{\beta}{\gamma}\mathrm{e}_\gamma(t,a) + \frac{\alpha}{\gamma}w(t) - w(a)
\end{aligned}
$$

so that

$$
\begin{aligned}
y(t) &\le \alpha(t-a)+\beta - [\alpha(t-a)+\beta] + \beta\,\mathrm{e}_\gamma(t,a) - \frac{\alpha}{\gamma} + \frac{\alpha}{\gamma}\mathrm{e}_\gamma(t,a) \\
&= \left(\frac{\alpha}{\gamma}+\beta\right)\mathrm{e}_\gamma(t,a) - \frac{\alpha}{\gamma}
\end{aligned}
$$

shows the correctness of our claim. □

We conclude this section with a Bihari type inequality from [16]. To prove this inequality, we need the following comparison result, whose proof can be found in [16, Theorem 3.1].

Theorem 5.7. *Let $h : \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ be continuous and nondecreasing in the second variable. Suppose v and w satisfy the dynamic inequalities*

$$v^\Delta \le h(t,v) \quad \text{and} \quad w^\Delta \ge h(t,w).$$

Then $v(t_0)\le w(t_0)$ for some $t_0\in\mathbb{T}$ implies $v(t)\le w(t)$ for all $t\in\mathbb{T}$.

The Bihari type inequality, see [16, Theorem 4.2], now reads as follows.

Theorem 5.8 (Bihari's Inequality)**.** *Suppose that g is continuous and nondecreasing, p is rd-continuous and nonnegative, and y is rd-continuous. Let w be the solution of*

$$w^\Delta = p(t)g(w), \quad w(a) = \beta$$

and suppose there is a bijective function G with $(G \circ w)^\Delta = p$. Then

$$y(t) \leq \beta + \int_a^t p(\tau)g(y(\tau))\Delta\tau \quad \text{for all} \quad t \in \mathbb{T}$$

implies

$$y(t) \leq G^{-1}\left[G(\beta) + \int_a^t p(\tau)\Delta\tau\right] \quad \text{for all} \quad t \in \mathbb{T}.$$

Proof. We denote

$$v(t) = \beta + \int_a^t p(\tau)g(y(\tau))\Delta\tau.$$

Then v satisfies

$$v^\Delta(t) = p(t)g(y(t)), \quad v(a) = \beta.$$

Since $p(t) \geq 0$ and $y(t) \leq v(t)$, the monotonicity of g implies

$$v^\Delta(t) = p(t)g(y(t)) \leq p(t)g(v(t)),$$

and so v satisfies

$$v^\Delta \leq p(t)g(v), \quad v(a) = \beta.$$

Therefore, by Theorem 5.7, $v(t) \leq w(t)$. Next,

$$G(w(t)) - G(\beta) = (G \circ w)(t) - (G \circ w)(a) = \int_a^t (G \circ w)^\Delta(\tau)\Delta\tau = \int_a^t p(\tau)\Delta\tau$$

so that

$$w(t) = G^{-1}\left[G(\beta) + \int_a^t p(\tau)\Delta\tau\right].$$

But $y(t) \leq v(t) \leq w(t)$ now shows the correctness of our claim. $\square$

6. Opial's Inequality

Opial inequalities and many of their generalizations have various applications in the theories of differential and difference equations. This is very nicely illustrated in the book [4] "Opial Inequalities with Applications in Differential and Difference Equations" by Agarwal and Pang, which is the only book devoted solely to continuous and discrete versions of Opial inequalities. In this section, following [7], we present several Opial inequalities that are valid on time scales. Throughout we assume $0 \in \mathbb{T}$ and let $h \in \mathbb{T}$ with $h > 0$.

We will need two simple consequences of the product rule: First,

$$(f^2)^\Delta = (f \cdot f)^\Delta = f^\Delta f + f^\sigma f^\Delta = (f + f^\sigma) f^\Delta, \tag{6.1}$$

and in general, one can use mathematical induction to prove the formula

$$(f^{l+1})^\Delta = \left\{ \sum_{k=0}^{l} f^k (f^\sigma)^{l-k} \right\} f^\Delta, \quad l \in \mathbb{N}. \tag{6.2}$$

Theorem 6.1 (Opial's Inequality)**.** *For a differentiable $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h |(x + x^\sigma) x^\Delta|(t) \Delta t \le h \int_0^h |x^\Delta|^2(t) \Delta t,$$

with equality when $x(t) = ct$.

Proof. Consider

$$y(t) = \int_0^t |x^\Delta(s)| \Delta s.$$

Then we have $y^\Delta = |x^\Delta|$ and $|x| \le y$ so that

$$\begin{aligned}
\int_0^h |(x+x^\sigma)x^\Delta|(t)\Delta t &\le \int_0^h [(|x|+|x^\sigma|)|x^\Delta|](t)\Delta t \\
&\le \int_0^h [(y+y^\sigma)|x^\Delta|](t)\Delta t \\
&= \int_0^h [(y+y^\sigma)y^\Delta](t)\Delta t \\
&= \int_0^h (y^2)^\Delta(t)\Delta t \\
&= y^2(h) - y^2(0) \\
&= \left\{\int_0^h |x^\Delta(t)|\Delta t\right\}^2 \\
&\le h\int_0^h |x^\Delta|^2(t)\Delta t,
\end{aligned}$$

where we have used formula (6.1) and Theorem 3.1 for $p = 1/2$.

Now, let $\tilde{x}(t) = ct$ for some $c \in \mathbb{R}$. Then $\tilde{x}^\Delta(t) \equiv c$, and it is easy to check that the equation

$$\int_0^h |(\tilde{x}+\tilde{x}^\sigma)\tilde{x}^\Delta|(t)\Delta t = h\int_0^h |\tilde{x}|^2(t)\Delta t$$

holds. □

We next state a generalization of Theorem 6.1 where $x(0)$ need not be equal to 0.

Theorem 6.2. *Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be differentiable. Then*

$$\int_0^h |(x+x^\sigma)x^\Delta|(t)\Delta t \le \alpha\int_0^h |x^\Delta(t)|^2\Delta t + 2\beta\int_0^h |x^\Delta(t)|\Delta t,$$

where

$$\alpha \in \mathbb{T} \quad \textit{with} \quad \mathrm{dist}(h/2, \alpha) = \mathrm{dist}(h/2, \mathbb{T}) \tag{6.3}$$

and $\beta = \max\{|x(0)|, |x(h)|\}$.

A consequence of Theorem 6.2 is the following result.

Theorem 6.3. *Let $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ be differentiable with $x(0) = x(h) = 0$. Then*

$$\int_0^h |(x + x^\sigma)x^\Delta|(t)\Delta t \le \alpha \int_0^h |x^\Delta(t)|^2 \Delta t,$$

where α is given in (6.3).

Now we offer some of the possible generalizations of the inequalities presented above. The continuous and/or discrete versions of these results may be found in [4]. We have not included all of such results, but most of them may be proved by using similar techniques as the ones presented in this section.

Theorem 6.4 (see [4, Theorem 2.5.1])**.** *Let p, q be positive and continuous on $[0, h]$, $\int_0^h \Delta t/p(t) < \infty$, and q non-increasing. For a differentiable $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h [q^\sigma |(x + x^\sigma)x^\Delta|](t)\Delta t \le \left\{ \int_0^h \frac{\Delta t}{p(t)} \right\} \left\{ \int_0^h p(t)q(t)|x^\Delta(t)|^2 \Delta t \right\}.$$

Theorem 6.5 (see [4, Chapter 3])**.** *Suppose $l, n \in \mathbb{N}$. For a n-times differentiable $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = x^\Delta(0) = \ldots = x^{\Delta^{n-1}}(0) = 0$ we have*

$$\int_0^h \left| \left\{ \sum_{k=0}^{l} x^k (x^\sigma)^{l-k} \right\} x^{\Delta^n} \right| (t)\Delta t \le h^{ln} \int_0^h \left| x^{\Delta^n}(t) \right|^{l+1} \Delta t.$$

Theorem 6.6 (see [4, Theorem 3.2.1])**.** *Suppose $n \in \mathbb{N}$. For a n-times differentiable $x : [0, h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = x^\Delta(0) = \ldots = x^{\Delta^{n-1}}(0) = 0$ we have*

$$\int_0^h |(x + x^\sigma)x^{\Delta^n}|(t)\Delta t \le h^n \int_0^h \left| x^{\Delta^n}(t) \right|^2 \Delta t.$$

Theorem 6.7 (see [4, Theorem 2.3.1])**.** *Suppose $l \in \mathbb{N}$. For a differentiable $x : [0,h] \cap \mathbb{T} \to \mathbb{R}$ with $x(0) = 0$ we have*

$$\int_0^h \left| \left\{ \sum_{k=0}^{l} x^k (x^\sigma)^{l-k} \right\} x^\Delta \right| (t) \Delta t \le h^l \int_0^h \left| x^\Delta(t) \right|^{l+1} \Delta t.$$

We conclude this section with a Wirtinger type inequality from [15].

Theorem 6.8 (Wirtinger's Inequality)**.** *Let M be positive and strictly monotone such that M^Δ exists and is rd-continuous. Then we have*

$$\int_a^b |M^\Delta(t)| (y^\sigma(t))^2 \Delta t \le \Psi \int_a^b \frac{M(t) M^\sigma(t)}{|M^\Delta(t)|} (y^\Delta(t))^2 \Delta t$$

for any y with $y(a) = y(b) = 0$ and such that y^Δ exists and is rd-continuous, where

$$\Psi = \left\{ \left(\sup_{t \in [a,b] \cap \mathbb{T}} \frac{M(t)}{M^\sigma(t)} \right)^{\frac{1}{2}} + \left[\left(\sup_{t \in [a,b] \cap \mathbb{T}} \frac{\mu(t) |M^\Delta(t)|}{M^\sigma(t)} \right) + \left(\sup_{t \in [a,b] \cap \mathbb{T}} \frac{M(t)}{M^\sigma(t)} \right) \right]^{\frac{1}{2}} \right\}^2.$$

Proof. For convenience we skip the argument (t) in this proof. Let

$$A = \int_a^b |M^\Delta| (y^\sigma)^2 \Delta t, \quad B = \int_a^b \frac{M M^\sigma}{|M^\Delta|} (y^\Delta)^2 \Delta t,$$
$$\alpha = \left(\sup_{t \in [a,b] \cap \mathbb{T}} \frac{M(t)}{M^\sigma(t)} \right)^{\frac{1}{2}}, \quad \beta = \left(\sup_{t \in [a,b] \cap \mathbb{T}} \frac{\mu(t) |M^\Delta(t)|}{M^\sigma(t)} \right).$$

Without loss of generality we assume that M^Δ is of positive sign. Then we apply the Cauchy-Schwarz inequality, Theorem 3.2, to estimate

$$\begin{aligned}
A &= \int_a^b M^\Delta (y^\sigma)^2 \Delta t \\
&= \int_a^b \left[(My^2)^\Delta - My^\Delta (y+y^\sigma)\right] \Delta t \\
&= -\int_a^b My^\Delta (y+y^\sigma)\Delta t \\
&\le \int_a^b M|y^\Delta||y+y^\sigma|\Delta t \\
&= \int_a^b M|y^\Delta||2y^\sigma - \mu y^\Delta|\Delta t \\
&\le 2\int_a^b M|y^\Delta||y^\sigma|\Delta t + \int_a^b \mu M (y^\Delta)^2 \Delta t \\
&= 2\int_a^b \sqrt{\frac{MM^\sigma}{|M^\Delta|}}|y^\Delta|\sqrt{\frac{M}{M^\sigma}|M^\Delta|}|y^\sigma|\Delta t + \int_a^b \frac{\mu M^\Delta}{M^\sigma}\frac{MM^\sigma}{|M^\Delta|}(y^\Delta)^2\Delta t \\
&\le 2\left\{\int_a^b \frac{MM^\sigma}{|M^\Delta|}(y^\Delta)^2\Delta t\right\}^{\frac{1}{2}}\left\{\int_a^b \frac{M}{M^\sigma}|M^\Delta|(y^\sigma)^2\Delta t\right\}^{\frac{1}{2}} + \beta B \\
&\le 2\alpha\sqrt{AB} + \beta B.
\end{aligned}$$

Therefore, by denoting $C = \sqrt{AB}$, we find that $C^2 - 2\alpha C - \beta \le 0$, and solving for $C \ge 0$ we obtain

$$C^2 \le (\alpha + \sqrt{\alpha^2+\beta})^2 = \Psi$$

so that the proof is complete. $\square$

Example 6.1. Let $a > 0$ and

$$\Psi = \left\{\left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\sigma(t)}{t}\right)^{\frac{1}{2}} + \left[\left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\mu(t)}{t}\right) + \left(\sup_{t\in[a,b]\cap\mathbb{T}} \frac{\sigma(t)}{t}\right)\right]^{\frac{1}{2}}\right\}^2.$$

Then

$$\int_a^b (y^\Delta(t))^2 \Delta t \geq \frac{1}{\Psi} \int_a^b \frac{(y^\sigma(t))^2}{t\sigma(t)} \Delta t. \tag{6.4}$$

To show this we remark that $M(t) = \frac{1}{t}$ satisfies the assumptions of Theorem 6.8, and

$$\frac{M(\sigma(t)) - M(s)}{\sigma(t) - s} = \frac{1/\sigma(t) - 1/s}{\sigma(t) - s} = \frac{(s - \sigma(t))/(s\sigma(t))}{\sigma(t) - s} = -\frac{1}{s\sigma(t)}$$

implies

$$M^\Delta(t) = -\frac{1}{t\sigma(t)}.$$

Therefore

$$\frac{M(t)}{M^\sigma(t)} = \frac{\sigma(t)}{t}, \quad \frac{\mu(t)|M^\Delta(t)|}{M^\sigma(t)} = \frac{\mu(t)}{t}, \quad \text{and} \quad \frac{M(t)M^\sigma(t)}{|M^\Delta(t)|} = 1,$$

and (6.4) follows from Theorem 6.8.

As an application of Theorem 6.8 and Example 6.1 we now state a sufficient criterion for nonoscillation of a certain second order dynamic equation, see [15, Theorem 3].

Theorem 6.9. *Let $N \in \mathbb{T}$ and define*

$$\Psi_N = \left\{ \left(\sup_{t \geq N, t \in \mathbb{T}} \frac{\sigma(t)}{t} \right)^{\frac{1}{2}} + \left[\left(\sup_{t \geq N, t \in \mathbb{T}} \frac{\mu(t)}{t} \right) + \left(\sup_{t \geq N, t \in \mathbb{T}} \frac{\sigma(t)}{t} \right) \right]^{\frac{1}{2}} \right\}^2.$$

If

$$0 < \limsup_{N \to \infty} \Psi_N = \Psi < \infty,$$

then the equation

$$y^{\Delta\Delta} + \frac{1}{\Psi t\sigma(t)} y^\sigma = 0$$

is nonoscillatory.

7. Lyapunov's Inequality

Lyapunov inequalities have proved to be very useful in oscillation theory, disconjugacy, eigenvalue problems, and numerous other applications in the theory of differential and difference equations. A nice summary of continuous and discrete Lyapunov inequalities and their applications can be found in the survey paper [10] by Chen. In this section we present several versions of Lyapunov inequalities on time scales. The established results supplement those presented in [6]. Throughout we assume $a, b \in \mathbb{T}$ with $a < b$.

We let $\mathbb{T} \subset \mathbb{R}$ be any time scale, $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous with $p(t) > 0$ for all $t \in \mathbb{T}$, and consider the Sturm-Liouville dynamic equation together with the quadratic functional

$$\mathcal{F}(x) = \int_a^b \left\{ (x^\Delta)^2 - p(x^\sigma)^2 \right\} (t) \Delta t.$$

To prove a Lyapunov inequality for

$$x^{\Delta^2} + p(t)x^\sigma = 0 \tag{7.1}$$

we need the following auxiliary results.

Lemma 7.1. *If x solves* (7.1) *and $\mathcal{F}(y)$ is defined, then*

$$\mathcal{F}(y) - \mathcal{F}(x) = \mathcal{F}(y - x) + 2(y - x)(b)x^\Delta(b) - 2(y - x)(a)x^\Delta(a).$$

Proof. Under the above assumptions we find

$$\begin{aligned}
\mathcal{F}(y)-\mathcal{F}(x)-\mathcal{F}(y-x) &= \int_a^b \big\{(y^\Delta)^2 - p(y^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2 \\
&\qquad -(y^\Delta - x^\Delta)^2 + p(y^\sigma - x^\sigma)^2\big\}(t)\Delta t \\
&= \int_a^b \big\{(y^\Delta)^2 - p(y^\sigma)^2 - (x^\Delta)^2 + p(x^\sigma)^2 - (y^\Delta)^2 + 2y^\Delta x^\Delta - (x^\Delta)^2 \\
&\qquad + p(y^\sigma)^2 - 2py^\sigma x^\sigma + p(x^\sigma)^2\big\}(t)\Delta t \\
&= 2\int_a^b \big\{y^\Delta x^\Delta - py^\sigma x^\sigma + p(x^\sigma)^2 - (x^\Delta)^2\big\}(t)\Delta t \\
&= 2\int_a^b \Big\{y^\Delta x^\Delta + y^\sigma x^{\Delta^2} - x^\sigma x^{\Delta^2} - (x^\Delta)^2\Big\}(t)\Delta t \\
&= 2\int_a^b \big\{yx^\Delta - xx^\Delta\big\}^\Delta \Delta t \\
&= 2\int_a^b \big\{(y-x)x^\Delta\big\}^\Delta \Delta t \\
&= 2(y(b)-x(b))x^\Delta(b) - 2(y(a)-x(a))x^\Delta(a),
\end{aligned}$$

where we have used the product rule (2.2). □

Lemma 7.2. *If $\mathcal{F}(y)$ is defined, then for any $r, s \in \mathbb{T}$ with $a \le r < s \le b$*

$$\int_r^s (y^\Delta(t))^2 \Delta t \ge \frac{(y(s)-y(r))^2}{s-r}.$$

Proof. Under the above assumptions we define

$$x(t) = \frac{y(s)-y(r)}{s-r}t + \frac{sy(r)-ry(s)}{s-r}.$$

We then have

$$x(r) = y(r), \quad x(s) = y(s), \quad x^\Delta(t) = \frac{y(s)-y(r)}{s-r}, \quad \text{and} \quad x^{\Delta^2}(t) = 0.$$

Hence x solves the special Sturm-Liouville equation (7.1) where $p = 0$ and therefore we may apply Lemma 7.1 to $\mathcal{F}_0$ defined by

$$\mathcal{F}_0(x) = \int_r^s (x^\Delta)^2(t)\Delta t$$

to find

$$\begin{aligned}
\mathcal{F}_0(y) &= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) + (y-x)(s)x^\Delta(s) - (y-x)(r)x^\Delta(r) \\
&= \mathcal{F}_0(x) + \mathcal{F}_0(y-x) \\
&\geq \mathcal{F}_0(x) \\
&= \int_r^s \left\{ \frac{y(s)-y(r)}{s-r} \right\}^2 \Delta t \\
&= \frac{(y(s)-y(r))^2}{s-r},
\end{aligned}$$

and this proves our claim. □

Using the above Lemma 7.2, we can now prove one of the main results of this section, a Lyapunov inequality for Sturm-Liouville dynamic equations of the form (7.1).

Theorem 7.1 (Lyapunov's Inequality)**.** *Let $p : \mathbb{T} \to \mathbb{R}_+$ be positive-valued and rd-continuous. If the Sturm-Liouville dynamic equation* (7.1) *has a nontrivial solution x with $x(a) = x(b) = 0$, then the Lyapunov inequality*

$$\int_a^b p(t)\Delta t \geq \frac{b-a}{f(d)}, \tag{7.2}$$

holds, where $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = (t-a)(b-t)$, and $d \in \mathbb{T}$ is such that

$$\left| \frac{a+b}{2} - d \right| = \min \left\{ \left| \frac{a+b}{2} - s \right| : \ s \in [a,b] \cap \mathbb{T} \right\}.$$

Proof. Suppose x is a solution of (7.1) with $x(a) = x(b) = 0$. Then we have from Lemma 7.1 (with $y = 0$) that

$$\mathcal{F}(x) = \int_a^b \left\{ (x^\Delta)^2 - p(x^\sigma)^2 \right\}(t)\Delta t \;=\; 0.$$

Since x is nontrivial, we find that M defined by

$$M = \max\left\{ x^2(t) :\; t \in [a,b] \right\} \tag{7.3}$$

is positive. We now let $c \in [a,b]$ to be such that $x^2(c) = M$. Applying the above as well as Lemma 7.2 twice (once with $r = a$ and $s = c$, and a second time with $r = c$ and $s = b$) we find

$$\begin{aligned} M\int_a^b p(t)\Delta t \;&\geq\; \int_a^b \left\{ p(x^\sigma)^2 \right\}(t)\Delta t \\ &=\; \int_a^b (x^\Delta)^2(t)\Delta t \;=\; \int_a^c (x^\Delta)^2(t)\Delta t + \int_c^b (x^\Delta)^2(t)\Delta t \\ &\geq\; \frac{(x(c)-x(a))^2}{c-a} + \frac{(x(b)-x(c))^2}{b-c} \\ &=\; x^2(c)\left\{ \frac{1}{c-a} + \frac{1}{b-c} \right\} \\ &=\; M\frac{b-a}{f(c)} \;\geq\; M\frac{b-a}{f(d)}, \end{aligned}$$

where the last inequality holds because of $f(d) = \max\{f(t) :\; t \in [a,b]\}$. Hence, dividing by $M > 0$ yields the desired inequality. $\square$

Example 7.1. We shall discuss the two classical cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\min\left\{ \left| \frac{a+b}{2} - s \right| :\; s \in [a,b] \right\} = 0 \quad \text{so that} \quad d = \frac{a+b}{2}.$$

Hence $f(d) = \frac{(b-a)^2}{4}$ and the Lyapunov inequality from Theorem 7.1 reads

$$\int_a^b p(t)\mathrm{d}t \geq \frac{4}{b-a}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we consider two cases. First, if $a+b$ is even, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : \ s \in [a,b]\right\} = 0 \quad \text{so that} \quad d = \frac{a+b}{2}.$$

Hence $f(d) = \frac{(b-a)^2}{4}$ and the Lyapunov inequality reads

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}.$$

If $a+b$ is odd, then

$$\min\left\{\left|\frac{a+b}{2} - s\right| : \ s \in [a,b]\right\} = \frac{1}{2} \quad \text{so that} \quad d = \frac{a+b-1}{2}.$$

This time we have $f(d) = \frac{(b-a)^2-1}{4}$ and the Lyapunov inequality reads

$$\sum_{t=a}^{b-1} p(t) \geq \frac{4}{b-a}\left\{\frac{1}{1-\frac{1}{(b-a)^2}}\right\}.$$

As an application of Theorem 7.1 we now state a sufficient criterion for disconjugacy of (7.1), see [6, Theorem 3.6] and also [2].

Theorem 7.2 (Sufficient Condition for Disconjugacy of (7.1))**.** *If p satisfies*

$$\int_a^b p(t)\Delta t < \frac{b-a}{f(d)}, \tag{7.4}$$

then (7.1) *is disconjugate on* $[a,b]$.

Remark 7.1. Note that in both conditions (7.2) and (7.4) we could

$$\text{replace} \quad \frac{b-a}{f(d)} \quad \text{by} \quad \frac{4}{b-a},$$

and Theorems 7.1 and 7.2 would remain true. This is because for $a \le c \le b$ we have

$$\frac{1}{c-a}+\frac{1}{b-c}=\frac{(a+b-2c)^2}{(b-a)(c-a)(b-c)}+\frac{4}{b-a}\ge\frac{4}{b-a}.$$

In the remainder of this section we present corresponding results for the linear Hamiltonian dynamic system

$$x^\Delta = A(t)x^\sigma + B(t)u, \quad u^\Delta = -C(t)x - A^*(t)u \tag{7.5}$$

where A, B, and C are rd-continuous $n\times n$-matrix-valued functions on $\mathbb{T}$ such that $I-\mu(t)A(t)$ is invertible and $B(t)$ and $C(t)$ are positive semidefinite for all $t\in\mathbb{T}$. The corresponding quadratic functional is given by

$$\mathcal{F}(x,u)=\int_a^b \left\{u^*Bu-(x^\sigma)^*Cx^\sigma\right\}(t)\Delta t.$$

We denote by $W(\cdot,r)$ the unique solution of the initial value problem

$$W^\Delta=-A^*(t)W, \quad W(r)=I,$$

where $r\in[a,b]$ is given, i.e., $W(t,r)=e_{-A^*}(t,r)$. Note that W exists due to our assumption on the invertibility of $I-\mu A$. Observe that $W(t,r)\equiv I$ provided $A(t)\equiv 0$. Finally, let

$$F(s,r)=\int_r^s W^*(t,r)B(t)W(t,r)\Delta t.$$

Theorem 7.3 (Lyapunov's Inequality)**.** *Assume* (7.5) *has a solution* (x,u) *such that* x *is nontrivial and satisfies* $x(a)=x(b)=0$. *With* W *and* F *as above, suppose that* $F(b,c)$ *and* $F(c,a)$ *are invertible, where* $||x(c)||=\max_{t\in[a,b]\cap\mathbb{T}}||x(t)||$. *Let* λ *be the largest eigenvalue of*

$$F=\int_a^b W^*(t,c)B(t)W(t,c)\Delta t,$$

and let $\nu(t)$ be the largest eigenvalue of $C(t)$. Then the Lyapunov inequality

$$\int_a^b \nu(t)\Delta t \geq \frac{4}{\lambda}$$

holds.

Remark 7.2. If $A \equiv 0$, then $W \equiv I$ and $F = \int_a^b B(t)\Delta t$. If, in addition $B \equiv 1$, then $F = b - a$. Note how the Lyapunov inequality $\int_a^b \nu(t)\Delta t \geq \frac{4}{\lambda}$ reduces to $\int_a^b p(t)\Delta t \geq \frac{4}{b-a}$ for the scalar case as discussed earlier in this section.

It is possible to provide a slightly better bound than the one given in Theorem 7.3, similarly as in Theorem 7.1, but we shall not do so here. Instead we now give a disconjugacy criterion for the system (7.5) whose proof is similar to that of Theorem 7.2.

Theorem 7.4 (Sufficient Condition for Disconjugacy of (7.5))**.** *Using notation from Theorem 7.3, if*

$$\int_a^b \nu(t)\Delta t < \frac{4}{\lambda},$$

then (7.5) *is disconjugate on* $[a, b]$*.*

We conclude this section with a result concerning so-called *right-focal* boundary conditions, i.e., $x(a) = u(b) = 0$.

Theorem 7.5. *Assume* (7.5) *has a solution* (x, u) *with* x *nontrivial and* $x(a) = u(b) = 0$*. With the notation as in Theorem 7.3, the Lyapunov inequality*

$$\int_a^b \nu(t)\Delta t \geq \frac{1}{\lambda}$$

holds.

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