ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO BOUNDARY VALUE PROBLEMS ON TIME SCALES

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ABSTRACT. This work formulates existence, uniqueness and uniqueness - implies - existence theorems for solutions to two-point vector boundary value problems on time scales. The methods used include maximum principles, a priori bounds on solutions and the nonlinear alternative of Leray-Schauder.

1. INTRODUCTION

This paper considers the existence and uniqueness of solutions to the second-order vector dynamic equation

$$(1) \qquad y^{\Delta\Delta}(t) = f(t, y(\sigma(t))) + P(t)y^{\Delta}(\sigma(t)), \quad t \in [a, b],$$

subject to any of the boundary conditions

$$(2) \qquad y(a) = A, \quad y(\sigma^2(b)) = B,$$

$$(3) \qquad \alpha y(a) - \beta y^{\Delta}(a) = C, \quad \gamma y(\sigma^2(b)) + \delta y^{\Delta}(\sigma(b)) = D,$$

$$(4) \qquad \alpha y(a) - \beta y^{\Delta}(a) = C, \quad y(\sigma^2(b)) = B,$$

$$(5) \qquad y(a) = A, \quad \gamma y(\sigma^2(b)) + \delta y^{\Delta}(\sigma(b)) = D,$$

where $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$; $P(t)$ is a $d \times d$ matrix; A, B, C, $D \in \mathbb{R}^d$ and α, β, γ, $\delta \in \mathbb{R}$. The problems (1), (2); (1), (3); (1), (4) and (1), (5) are known as boundary value problems (BVPs) on "time scales".

To understand the notation used above and the idea of time scales some preliminary definitions are useful.

Definition A time scale $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since a time scale may or may not be connected, the concept of jump operators is useful.

Definition Define the forward (backward) jump operator $\sigma(t)$ at t for $t < \sup \mathbb{T}$ (respectively $\rho(t)$ at t for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\}, \quad (\rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\},) \text{ for all } t \in \mathbb{T}.$$

Also define $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} < \infty$, and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} > -\infty$. For simplicity and clarity denote $\sigma^2(t) = \sigma(\sigma(t))$ and $y^\sigma(t) = y(\sigma(t))$. Define the graininess function $\mu : \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$.

Throughout this work the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. Also assume throughout that $a < b$ are points in $\mathbb{T}$ with $[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}$.

The jump operators σ and ρ allow the classification of points in a time scale in the following way: If $\sigma(t) > t$ then call the point t right-scattered; while if $\rho(t) < t$ then we say t is left-scattered. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then call the point t right-dense; while if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then we say t is left-dense.

If $\mathbb{T}$ has a left-scattered maximum at m then define $\mathbb{T}^k = \mathbb{T} - \{m\}$. Otherwise $\mathbb{T}^k = \mathbb{T}$.

Definition Fix $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}^d$. Define $y^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t such that, for all $s \in U$ and each $i = 1, \ldots, d$,

$$|[y_i(\sigma(t)) - y_i(s)] - y_i^\Delta(t)[\sigma(t) - s]| \leq \epsilon |\sigma(t) - s|.$$

Call $y^\Delta(t)$ the (delta) derivative of $y(t)$ at t.

Definition If $F^\Delta(t) = f(t)$ then define the integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

The following theorem is due to Hilger [12].

Theorem 1. *Assume that $f : \mathbb{T} \to \mathbb{R}^d$ and let $t \in \mathbb{T}^k$.*

(i) If f is differentiable at t then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) If f is differentiable and t is right-dense then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f is differentiable at t then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.

Definition Define $f \in C_{rd}(\mathbb{T}; \mathbb{R}^d)$ as right-dense continuous if at all $t \in \mathbb{T}$ then:

(a) f is continuous at every right-dense point $t \in \mathbb{T}$, and

(b) $\lim_{s \to t^-} f(s)$ exists and is finite at every left-dense point $t \in \mathbb{T}$.

Definition Define S to be the set of all functions $y : \mathbb{T} \to \mathbb{R}^d$ such that

$$S = \{y : y \in C([a, \sigma(b)]; \mathbb{R}^d) \text{ and } y^{\Delta\Delta} \in C_{rd}([a, b]; \mathbb{R}^d)\}.$$

A solution to (1) is a function $y \in S$ which satisfies (1) for each $t \in [a, b]$.

In order to prove the existence of solutions to the BVPs (1), (2) through (1), (5) the following theorem shall be used, which is referred to as the nonlinear alternative of Leray-Schauder.

Theorem 2. *Let Ω be an open, bounded subset of a Banach space X with $0 \in \Omega$ and let $T : \bar{\Omega} \to X$ be a compact operator. If $y \neq \lambda T(y)$ for all $y \in \partial\Omega$ and all $\lambda \in [0, 1]$ then $y = T(y)$ for some $y \in \Omega$.*

Proof This is a special case of Lloyd [15, Theorem 4.4.11].

Recently the study of dynamic equations on time scales has attracted much interest (see [1] - [4], [6] - [10], [12] - [14]). This has been mainly due to its unification of the theory of differential and difference equations. The potential for applications is enormous - especially in those phemenona that manifest themselves partly in continuous time and partly in discrete time.

To the authors' knowledge no papers have yet dealt with second-order systems of BVPs on time scales. The extension to systems is a natural one; for example, many occurances in nature involve two or more populations co-existing in an environment,

with the model being best described by a system of dynamic equations. (Beltrami [5] Section 5.6 discusses algae and copepod populations via second-order systems of BVPs.)

This paper deals with two specific types of second-order equations. Sections 2-6 treat the nonlinear equation,

$$(6) \qquad\qquad y^{\Delta\Delta}(t) = f(t, y^{\sigma}(t)), \quad t \in [a, b],$$

and Section 7 treats the linear equation,

$$(7) \qquad\qquad y^{\Delta\Delta}(t) = P(t)y^{\Delta}(\sigma(t)) + Q(t)y^{\sigma}(t) + h(t), \quad t \in [a, b],$$

where P and Q are $d \times d$ matrices functions and h is a $d \times 1$ vector function.

In particular, the paper is organized as follows.

In Section 2, the necessary a priori bounds on solutions to the BVPs (6), (2) through (6), (5) are formulated via some simple lemmas involving inequalities on f and on the boundary conditions.

In Section 3, the a priori bounds from Section 2 are used in conjunction with the nonlinear alternative to prove the existence of solutions to the BVPs (6), (2) through (6), (5).

In Section 4, the inequalities on f from Section 3 are slightly strenthened and some extra qualitative information about solutions is obtained. Solutions are shown to be nonincreasing or nondecreasing in norm.

In Section 5, BVPs on infinite intervals are investigated and some existence theorems are presented. The proofs rely on the existence of solutions on finite intervals and so use the theorems of Section 3. A standard diagonalization argument is also employed.

In Sections 6 and 7, some simple maximum principles are used to prove the uniqueness of solutions to (6), (2) and (7), (2). A simple uniqueness-implies-existence theorem is also presented for (7), (2).

The theory of time scales dates back to Hilger [12]. The monographs [6] and [14] also provide an excellent introduction. Of particular motivation for the research in this paper were the works [1] - [4] and [8] - [10].

2. A Priori Bounds on Solutions

In order to apply Theorem 2, a priori bounds on solutions to the BVPs are needed. In this section conditions on f and on the boundary conditions are formulated under which these bounds are guaranteed.

The following maximum principle shall be very useful throughout the rest of the paper and can be found in Gnana Bhaskar [10].

Lemma 1. *If a function $r : \mathbb{T} \to \mathbb{R}$ has a local maximum at a point $c \in [a, \sigma^2(b)]$ then $r^{\Delta\Delta}(\rho(c)) \leq 0$ provided c is not simultaneously left-dense and right-scattered and that $r^{\Delta\Delta}(\rho(c))$ exists.*

Let y be a solution to (1). In what follows below, the maximum principle of Lemma 1 shall be applied to the "Lyapunov-type" function $r(t) = \|y(t)\|^2$ and then used to show that r is bounded on $[a, \sigma^2(b)]$ (and therefore solutions y are bounded on $[a, \sigma^2(b)]$). In order to guarantee that $r^{\Delta\Delta}(\rho(c))$ exists, both $y^{\Delta\Delta}(t)$ and $[y(\sigma(t))]^{\Delta}$ must exist, since $r(t) = \langle y(t), y(t) \rangle$ is the inner product of two functions. As remarked in [6], the product of two functions is not necessarilly differentiable even if each of the functions are twice differentiable. Therefore for the remainder of the paper, assume that $\sigma(t)$ is such that for those solutions $y \in S$, $[y(\sigma(t))]^{\Delta}$ exists.

Lemma 2. *Let $R > 0$ be a constant such that*

(8) $$\langle u, f(t, u) \rangle > 0, \text{ for all } t \in [a, b] \text{ and } \|u\| \geq R.$$

If y is a solution to (6) and $\|y(t)\|$ does not achieve its maximum value at $t = a$ or $t = \sigma^2(b)$, then $\|y(t)\| < R$ for $t \in [a, \sigma^2(b)]$.

Proof Assume that the conclusion of the lemma is false. Therefore $r(t) := \|y(t)\|^2 - R^2$ must have a nonnegative maximum in $[a, \sigma^2(b)]$. By hypothesis, this maximum must occur in $(a, \sigma^2(b))$. Choose $c \in (a, \sigma^2(b))$ such that

$$r(c) = \max\{r(t); t \in [a, \sigma^2(b)]\} \geq 0,$$

and

$$(9) \qquad\qquad r(t) < r(c), \quad \text{for } c < t < \sigma^2(b).$$

First we show that the point c cannot be simultaneously left-dense and right-scattered. Assume the contrary by letting $\rho(c) = c < \sigma(c)$. If $r^\Delta(c) \geq 0$ then $r(\sigma(c)) \geq r(c)$ and this contradicts (9). If $r^\Delta(c) < 0$ then $\lim_{t \to c^-} r^\Delta(t) = r^\Delta(c) < 0$. Therefore there exists a $\delta > 0$ such that $r^\Delta(t) < 0$ on $(c - \delta, c]$. Hence $r(t)$ is strictly decreasing on $(c - \delta, c]$ and this contradicts the way c was chosen.

Therefore the point c cannot be simultaneously left-dense and right-scattered.

By Lemma 1 we must have

$$(10) \qquad\qquad r^{\Delta\Delta}(\rho(c)) \leq 0.$$

So using the product rule (see [6]) we have

$$
\begin{aligned}
r^{\Delta\Delta}(\rho(c)) &= 2\langle y^\sigma(\rho(c)), f(\rho(c), y^\sigma(\rho(c)))\rangle + \|y^\Delta(\rho(c))\|^2 + \|y^{\sigma\Delta}(\rho(c))\|^2, \\
&\geq 2\langle y^\sigma(\rho(c)), f(\rho(c), y^\sigma(\rho(c)))\rangle > 0, \quad \text{by (8)},
\end{aligned}
$$

which contradicts (10). Therefore $\|y(t)\| < R$ for $t \in [a, \sigma^2(b)]$. (Notice at c that $\|y^\sigma(\rho(c))\| = \|y(c)\| \geq R$, since c is not simultaneously left-dense and right-scattered.) This concludes the proof.

The following lemma provides a priori bounds on solutions to the Dirichlet BVP (6), (2).

Lemma 3. *If f and R satisfy the conditions of Lemma 2 with $\|A\|$, $\|B\| < R$, then every solution y to the Dirichlet BVP (6), (2) satisfies $\|y(t)\| < R$ for $t \in [a, \sigma^2(b)]$.*

Proof This result follows immediately from Lemma 2.

The following lemma provides a priori bounds on solutions to the Sturm-Liouville BVP (6), (3).

Lemma 4. *If f and R satisfy the conditions of Lemma 2 with α, β, γ, $\delta > 0$, then every solution y to the Sturm-Liouville BVP (6), (3) satisfies*

$$\|y(t)\| < \max\{\|C\|/\alpha, \ \|D\|/\gamma, \ R\} + 1, \quad \text{for } t \in [a, \sigma^2(b)].$$

Proof Let $M = \max\{\|C\|/\alpha, \ \|D\|/\gamma, \ R\}$ and assume that $r(t) := \|y(t)\|^2 - (M+1)^2$ has a nonnegative maximum at $t = a$. Then

$$
\begin{aligned}
r^\Delta(a) &= \langle y(a) + y^\sigma(a), y^\Delta(a) \rangle \\
&= \langle 2y(a) + \mu(a)y^\Delta(a), y^\Delta(a) \rangle \\
&= 2\langle y(a), y^\Delta(a) \rangle + \mu(a)\|y^\Delta(a)\|^2 \\
&\leq 0.
\end{aligned}
$$

It follows that

$$
2\langle y(a), y^\Delta(a) \rangle \leq -\mu(a)\|y^\Delta(a)\|^2 \leq 0
$$

and therefore

$$
\langle y(a), y^\Delta(a) \rangle \leq 0.
$$

Hence

$$
0 \geq \langle y(a), \beta y^\Delta(a) \rangle = \langle y(a), \alpha y(a) - C \rangle = \alpha\|y(a)\|^2 \left(1 - \frac{\langle y(a), C \rangle}{\alpha\|y(a)\|^2}\right),
$$

and therefore $\left(1 - \frac{\langle y(a), C \rangle}{\alpha\|y(a)\|^2}\right) \leq 0$. Hence we have

$$
(11) \qquad 1 \leq \frac{\langle y(a), C \rangle}{\alpha\|y(a)\|^2} \leq \frac{|\langle y(a), C \rangle|}{\alpha\|y(a)\|^2} \leq \frac{\|y(a)\|\|C\|}{\alpha\|y(a)\|^2} = \frac{\|C\|}{\alpha\|y(a)\|}.
$$

Thus rearranging (11) we obtain $\|y(a)\| \leq \|C\|/\alpha \leq M$. If a nonnegative maximum occurs at $t = \sigma^2(b)$, then

$$
\begin{aligned}
r^\Delta(\sigma(b)) &= \langle y(\sigma(b)) + y^\sigma(\sigma(b)), y^\Delta(\sigma(b)) \rangle \\
&= \langle 2y^\sigma(\sigma(b)) - \mu(\sigma(b))y^\Delta(\sigma(b)), y^\Delta(\sigma(b)) \rangle \\
&= 2\langle y(\sigma^2(b)), y^\Delta(\sigma(b)) \rangle - \mu(\sigma(b))\|y^\Delta(\sigma(b))\|^2 \\
&\geq 0.
\end{aligned}
$$

It follows that

$$
2\langle y(\sigma^2(b)), y^\Delta(\sigma(b)) \rangle \geq \mu(\sigma(b))\|y^\Delta(\sigma(b))\|^2 \geq 0
$$

and therefore

$$
\langle y(\sigma^2(b)), y^\Delta(\sigma(b)) \rangle \geq 0.
$$

Hence

$$
\begin{aligned}
0 \;\; &\leq \;\; \langle y(\sigma^2(b)), \delta y^\Delta(\sigma(b)) \rangle \\
&= \;\; \langle y(\sigma^2(b)), D - \gamma y(\sigma^2(b)) \rangle \\
&= \;\; \gamma \| y(\sigma^2(b)) \|^2 \left(\frac{\langle y(\sigma^2(b)), D \rangle}{\gamma \| y(\sigma^2(b)) \|^2} - 1 \right),
\end{aligned}
$$

and therefore $\left(\frac{\langle y(\sigma^2(b)), D \rangle}{\gamma \| y(\sigma^2(b)) \|^2} - 1 \right) \geq 0$. Hence we have

$$
(12) \qquad 1 \leq \frac{\langle y(\sigma^2(b)), D \rangle}{\gamma \| y(\sigma^2(b)) \|^2} \leq \frac{|\langle y(\sigma^2(b))), D \rangle|}{\gamma \| y(\sigma^2(b)) \|^2} \leq \frac{\| y(\sigma^2(b)) \| \| D \|}{\gamma \| y(\sigma^2(b)) \|^2} = \frac{\| D \|}{\gamma \| y(\sigma^2(b)) \|}.
$$

Thus rearranging (12) we obtain $\| y(\sigma^2(b)) \| \leq \| D \| / \gamma \leq M$. If a maximum occurs in $(a, \sigma^2(b))$ then $\| y(t) \| < R$, $t \in [a, \sigma^2(b)]$ by Lemma 2. This concludes the proof.

The question now arises on whether the conditions α, β, γ, $\delta > 0$ can be removed from Lemma 4. By "piecing together" parts of Lemmas 3 and 4, results for the BVPs (6), (4) and (6), (5) are now presented.

Lemma 5. *Let f and R satisfy the conditions of Lemma 2. If α, $\beta > 0$ and $\| B \| < R$, then every solution y to the BVP (6), (4), satisfies*

$$
\| y(t) \| < \max\{ \| C \| / \alpha, \; R \} + 1, \quad \text{for } t \in [a, \sigma^2(b)].
$$

Lemma 6. *Let f and R satisfy the conditions of Lemma 2. If γ, $\delta > 0$ and $\| A \| < R$, then every solution y to the BVP (6), (5) satisfies*

$$
\| y(t) \| < \max\{ \| D \| / \gamma, \; R \} + 1, \quad \text{for } t \in [a, \sigma^2(b)].
$$

Proofs The proofs follow similar lines to those of Lemmas 3 and 4 and so are omitted.

3. Existence of Solutions

In this section some existence results are presented for the BVPs (6), (2) through (6), (5). The proofs rely on the a priori bounds on solutions of Section 2 and the nonlinear alternative.

The following theorem gives the existence of solutions to the Dirichlet BVP on time scales.

Theorem 3. *Let $R > 0$ be a constant. Suppose that $f(t, u)$ is continuous on $[a, b] \times \mathbb{R}^d$ and satisfies (8). If $\|A\|$, $\|B\| < R$, then the Dirichlet BVP (6), (2) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a, \sigma^2(b)]$.*

Proof The BVP (6), (2) is equivalent (see Corollary 4.76 in [6]) to the integral equation

$$(13) \qquad y(t) = \int_a^{\sigma(b)} G(t, s) f(s, y^\sigma) \Delta s + \phi(t), \quad t \in [a, \sigma^2(b)],$$

where

$$G(t, s) = \begin{cases} \frac{(t-a)(\sigma^2(b)-\sigma(s))}{(\sigma^2(b)-a)}, & \text{for } t \leq s, \\[2mm] \frac{(\sigma(s)-a)(\sigma^2(b)-t)}{(\sigma^2(b)-a)}, & \text{for } \sigma(s) \leq t. \end{cases}$$

and

$$\phi(t) = \frac{A\sigma^2(b) - Ba + (B - A)t}{\sigma^2(b) - a}.$$

Thus we want to prove that there exists at least one y satisfying (13). Define an operator $T : C([a, \sigma^2(b)]; \mathbb{R}^d) \to C([a, \sigma^2(b)]; \mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t, s) f(s, y^\sigma(s)) \Delta s + \phi(t).$$

If we can prove that there exists a y such that $T(y) = y$, then there exists a solution to (13). To show that T has a fixed point, consider the equation

$$(14) \qquad y = \lambda T(y), \quad \text{for } \lambda \in [0, 1].$$

Define an open, bounded subset of the Banach space S by $\Omega = \{y \in S : \|y\| < R\}$, where here $\|\cdot\|$ is the sup norm. Note that (14) is equivalent to the BVP

$$(15) \qquad y^{\Delta\Delta}(t) = \lambda f(t, y(\sigma(t))), \quad t \in [a, b],$$

$$(16) \qquad y(a) = \lambda A, \quad y(\sigma^2(b)) = \lambda B.$$

Now show that all solutions to (15), (16) must satisfy $y \in \Omega$, and consequently $y \notin \partial\Omega$ for all $\lambda \in [0, 1]$. Obviously $y \in \Omega$ for $\lambda = 0$. So consider (15), (16) for $\lambda \in (0, 1]$. Note that by (8),

$$\langle u, \lambda f(t, u) \rangle = \lambda \langle u, f(t, u) \rangle > 0, \text{ for all } t \in [a, b] \text{ and } \|u\| \geq R.$$

Also $\|\lambda A\|$, $\|\lambda B\| \le \|A\|$, $\|B\| < R$. Therefore Lemma 3 is applicable to solutions of (15), (16). Hence all solutions y to (15), (16) must satisfy $\|y(t)\| < R$ for $t \in [a, \sigma^2(b)]$. Hence $y \notin \partial\Omega$.

Since f is continuous, T is continuous and it can be shown that T is a compact operator by the Arzela-Ascoli Theorem. Therefore Theorem 2 is applicable to T and T must have a fixed point. Hence the BVP has a solution. This concludes the proof.

The following theorem gives the existence of solutions to the Sturm-Liouville BVP on time scales.

Theorem 4. *Let $R > 0$ be a constant. Suppose that f is continuous on $[a, b] \times \mathbb{R}^d$ and satisfies*

$$(17) \qquad \langle u, f(t, u) \rangle > 0, \quad \text{for all } t \in [a, b] \text{ and } \|u\| \ge R.$$

If α, β, γ, $\delta > 0$, then the Sturm-Liouville BVP (6), (3) has at least one solution $y \in S$ satisfying

$$\|y(t)\| < \max\{\|C\|/\alpha, \ \|D\|/\gamma, \ R\} + 1, \quad \text{for } t \in [a, \sigma^2(b)].$$

Proof The BVP (6), (3) is equivalent to the integral equation

$$(18) \qquad y(t) = \int_a^{\sigma(b)} G(t, s) f(s, y^\sigma(s)) \Delta s + \phi(t), \quad t \in [a, \sigma^2(b)],$$

where

$$G(t, s) = \begin{cases} -\left[\beta + (t - a)\alpha\right]\left[\delta + (\sigma^2(b) - \sigma(s))\gamma\right]/p, & \text{for } t \le s, \\ -\left[\beta + (\sigma(s) - \alpha)a\right]\left[\delta + (\sigma^2(b) - t)\gamma\right]/p, & \text{for } \sigma(s) \le t. \end{cases}$$

and

$$p = \alpha\gamma(\sigma^2(b) - a) + \alpha\delta + \beta\gamma, \ \ \phi(t) = [(\gamma\sigma^2(b) + \delta)C + (\beta - \alpha a)D + (D\alpha - C\gamma)t]/p.$$

Thus we want to prove that there exists at least one y satisfying (18). Define an operator $T : C([a, \sigma^2(b)]; \mathbb{R}^d) \to C([a, \sigma^2(b)]; \mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t, s) f(s, y^\sigma(s)) \Delta s + \phi(t).$$

If we can prove that there exists a y such that $T(y) = y$, then there exists a solution to (18). To show that T has a fixed point, consider the equation

$$(19) \qquad y = \lambda T(y), \quad \text{for } \lambda \in [0, 1].$$

Define an open, bounded subset of the Banach space S by

$$\Omega = \{y \in S : \|y\| < \max\{\|C\|/\alpha, \ \|D\|/\gamma, \ R\} + 1\}.$$

Note that (19) is equivalent to the BVP

(20) $$y^{\Delta\Delta} = \lambda f(t, y^\sigma), \quad t \in [a, b],$$

(21) $$\alpha y(a) - \beta y^\Delta(a) = \lambda C, \qquad \gamma y(\sigma^2(b)) + \delta y^\Delta(\sigma(b)) = \lambda D.$$

Now show that all solutions to (20), (21) must satisfy $y \in \Omega$, and consequently $y \notin \partial\Omega$ for all $\lambda \in [0, 1]$. Obviously $y \in \Omega$ for $\lambda = 0$. So consider (20), (21) for $\lambda \in (0, 1]$. Note that

$$\langle u, \lambda f(t, u) \rangle = \lambda \langle u, f(t, u) \rangle > 0, \ \text{for all } t \in [a, b] \text{ and } \|u\| \geq R, \ \text{by (17)}.$$

Since $\lambda\alpha, \lambda\beta, \lambda\gamma, \lambda\delta > 0$, we get that Lemma 4 is applicable to solutions of (20), (21), and hence

$$\|y(t)\| \leq \max\{\|\lambda C\|/\alpha, \ \|\lambda D\|/\gamma, \ R\} \leq \max\{\|C\|/\alpha, \ \|D\|/\gamma, \ R\}$$

for $t \in [a, \sigma^2(b)]$. Hence all solutions y to (20), (21) must satisfy

$$\|y(t)\| < \max\{\|C\|/\alpha, \ \|D\|/\gamma, \ R\} + 1$$

for $t \in [a, \sigma^2(b)]$ and therefore $y \notin \partial\Omega$.

Since f is continuous, T is continuous and it can be shown that T is a compact operator by the Arzela-Ascoli Theorem. Therefore Theorem 2 is applicable to T and T must have a fixed point. Hence the BVP has a solution. This concludes the proof.

The following result gives the existence of solutions to the BVP (6), (4), and we will use this in Section 4 when dealing with BVPs on infinite intervals.

Theorem 5. *Let $R > 0$ be a constant. Suppose that f is continuous on $[a, b] \times \mathbb{R}^d$ and satisfies (8). If $\alpha, \beta > 0$ and $\|B\| < R$, then the BVP (6), (4) has at least one solution $y \in S$ satisfying $\|y(t)\| < \max\{\|C\|/\alpha, \ R\} + 1$ for $t \in [a, \sigma^2(b)]$.*

Proof The BVP (6), (4) is equivalent to the integral equation

(22) $$y(t) = \int_a^{\sigma(b)} G(t, s) f(s, y^\sigma(s)) \Delta s + \phi(t), \quad t \in [a, \sigma^2(b)],$$

where

$$G(t,s) = \begin{cases} -\left[\beta + (t-a)\alpha\right]\left[(\sigma^2(b) - \sigma(s))\right]/p, & \text{for } t \le s, \\ -\left[\beta + (\sigma(s) - \alpha)a\right]\left[(\sigma^2(b) - t)\right]/p, & \text{for } \sigma(s) \le t, \end{cases}$$

and

$$p = \alpha(\sigma^2(b) - a) + \beta, \quad \phi(t) = [\sigma^2(b)C + (\beta - \alpha a)B + (B\alpha - C)t]/p.$$

Thus we want to prove that there exists at least one y satisfying (22). Define an operator $T : C([a, \sigma^2(b)]; \mathbb{R}^d) \to C([a, \sigma^2(b)]; \mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t,s) f(s, y^\sigma(s)) \Delta s + \phi(t).$$

If we can prove that there exists a y such that $T(y) = y$, then there exists a solution to (18). To show that T has a fixed point, consider the equation

(23) $$y = \lambda T(y), \quad \text{for } \lambda \in [0, 1].$$

Define an open, bounded subset of the Banach space S by

$$\Omega = \{y \in S : \|y\| < \max\{\|C\|/\alpha, \ R\} + 1\}.$$

Note that (23) is equivalent to the BVP

(24) $$y^{\Delta\Delta} = \lambda f(t, y^\sigma), \quad t \in [a, b],$$

(25) $$\alpha y(a) - \beta y^\Delta(a) = \lambda C, \qquad y(\sigma^2(b)) = \lambda B.$$

Now show that all solutions to (24), (25) must satisfy $y \in \Omega$ and consequently $y \notin \partial\Omega$ for all $\lambda \in [0, 1]$. Obviously $y \in \Omega$ for $\lambda = 0$. So consider (24), (25) for $\lambda \in (0, 1]$. Note that

$$\langle u, \lambda f(t, u)\rangle = \lambda\langle u, f(t, u)\rangle > 0, \ \text{for all } t \in [a, b] \text{ and } \|u\| \ge R, \ \text{by (8)}.$$

Since $\lambda\alpha, \lambda\beta > 0$ and $\|\lambda B\| \le \|B\| < R$, we see that Lemma 5 is applicable to solutions of (24), (25), and hence

$$\|y\| \le \max\{\|\lambda C\|/\alpha, \ R\} \le \max\{\|C\|/\alpha, \ R\}.$$

Therefore all solutions y to (24), (25) must satisfy $\|y\| < \max\{\|C\|/\alpha, \ R\} + 1$ and $y \notin \partial\Omega$.

Since f is continuous, T is continuous and it can be shown that T is a compact operator by the Arzela-Ascoli Theorem. Therefore Theorem 2 is applicable to T and T must have a fixed point. Hence the BVP has a solution. This concludes the proof.

Similarly, the following result holds.

Theorem 6. *Let $R > 0$ be a constant. Suppose that f is continuous on $[a, b] \times \mathbb{R}^d$ and satisfies (8). If $\|A\| < R$ and γ, $\delta > 0$, then the BVP (1), (5) has at least one solution $y \in S$ satisfying*

$$\|y(t)\| < \max\{\|D\|/\gamma, \; R\} + 1, \quad \text{for } t \in [a, \sigma^2(b)].$$

Proof The proof is similar to that of Theorem 5 and so is omitted.

Remark 1. *Theorems 3 - 6 establish bounds on* all *solutions to the respective BVPs (1), (2) - (5). If there is no concern about bounding all of the solutions to the BVPs then inequality (8) may be weakened to*

(26) $$\langle u, f(t, u) \rangle > 0, \quad \text{for all } t \in [a, b] \text{ and } \|u\| = R,$$

and existence results will still hold, as the following theorems demonstrate.

Theorem 7. *Let the conditions of Theorem 3 hold with (8) replaced with (26). Then the Dirichlet BVP (6), (2) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a, \sigma^2(b)]$ (and there may exist further solutions satisfying $\|y(t_0)\| \geq R$ for some $t_0 \in [a, \sigma^2(b)]$).*

Proof Consider the modified dynamic equation

(27) $$y^{\Delta\Delta} = m(t, y^\sigma), \quad t \in [a, b],$$

subject to the boundary conditions (2), where

$$m(t, y^\sigma) = \begin{cases} Rf(t, Ry^\sigma/\|y^\sigma\|)/\|y^\sigma\|, & \text{for } \|y^\sigma\| \geq R, \\ f(t, y^\sigma), & \text{for } \|y^\sigma\| \leq R. \end{cases}$$

Similarly to the proof of Theorem 3, define an operator $T : C([a, \sigma^2(b)]; \mathbb{R}^d) \to C([a, \sigma^2(b)]; \mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t, s)m(s, y^\sigma(s))\Delta s + \phi(t),$$

where G and ϕ are given in the proof of Theorem 3. To show that T has a fixed point, consider the equation

$$y = \lambda T(y), \quad \text{for } \lambda \in [0, 1].$$

Define an open, bounded subset of the Banach space S by $\Omega = \{y \in S : \|y\| < ME + N + 1\}$, where here $\| \cdot \|$ is the sup norm, E is the bound on m and

$$
\begin{aligned}
M &= \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} |G(t, s)| \Delta s, \\
N &= \max_{t \in [a, \sigma^2(b)]} |\phi(t)|.
\end{aligned}
$$

It is easy to see that $\|\lambda T(y)\| \leq \lambda(ME + N) < ME + N + 1$ for all $\lambda \in [0, 1]$ and see that Theorem 2 is applicable. Therefore the BVP (27), (2) has a solution $y \in \Omega$. To show that this is a solution of the BVP (6), (2) see that, for $\|y\| \geq R$

$$\langle y^\sigma, m(t, y^\sigma) \rangle = \langle p, f(t, p) \rangle > 0,$$

for $\|y\| \geq R = \|p\|$ by (26) and $p = Ry^\sigma / \|y^\sigma\|$. Therefore all solutions to (27), (2) satisfy $\|y\| < R$ and are solutions to the BVP (6), (2). This concludes the proof.

Theorem 8. *Let the conditions of Theorem 4 hold with (8) replaced with (26) and* $\max\{\|C\|/\alpha, \|D\|/\beta\} < R$. *Then the Sturm-Liouville BVP (6), (3) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a, \sigma^2(b)]$ (and there may exist further solutions satisfying $\|y(t_0)\| \geq R$ for some $t_0 \in [a, \sigma^2(b)]$).*

Consider the modified BVP (27), (3). Similarly to the proof of Theorem 4, define an operator $T : C([a, \sigma^2(b)]; \mathbb{R}^d) \to C([a, \sigma^2(b)]; \mathbb{R}^d)$ by

$$(Ty)(t) = \int_a^{\sigma(b)} G(t, s) m(s, y^\sigma(s)) \Delta s + \phi(t),$$

where G and ϕ are given in the proof of Theorem 4. To show that T has a fixed point, consider the equation

$$y = \lambda T(y), \quad \text{for } \lambda \in [0, 1].$$

Define an open, bounded subset of the Banach space S by $\Omega = \{y \in S : \|y\| < ME + N + 1\}$, where here $\| \cdot \|$ is the sup norm, E is the bound on m and

$$M = \max_{t \in [a, \sigma^2(b)]} \int_a^{\sigma(b)} |G(t, s)| \Delta s,$$
$$N = \max_{t \in [a, \sigma^2(b)]} |\phi(t)|.$$

It is easy to see that $\|\lambda T(y)\| \leq \lambda(ME + N) < ME + N + 1$ for all $\lambda \in [0, 1]$ and see that Theorem 2 is applicable. Therefore the BVP (27), (3) has a solution $y \in \Omega$. To show that this is a solution of the BVP (6), (3) see that, for $\|y\| \geq R$

$$\langle y^\sigma, m(t, y^\sigma) \rangle = \langle p, f(t, p) \rangle > 0,$$

for $\|y\| \geq R = \|p\|$ by (26) and $p = Ry^\sigma / \|y^\sigma\|$. Therefore all solutions to (27), (3) satisfy $\|y\| < R$ and are solutions to the BVP (6), (3). This concludes the proof.

Theorem 9. *Let the conditions of Theorem 5 hold with (8) replaced with (26) and $\|C\|/\alpha < R$. Then the BVP (6), (4) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a, \sigma^2(b)]$ (and there may exist further solutions satisfying $\|y(t_0)\| \geq R$ for some $t_0 \in [a, \sigma^2(b)]$).*

Theorem 10. *Let the conditions of Theorem 6 hold with (8) replaced with (26) and $\|D\|/\beta < R$. Then the BVP (6), (5) has at least one solution $y \in S$ satisfying $\|y(t)\| < R$ on $[a, \sigma^2(b)]$ (and there may exist further solutions satisfying $\|y(t_0)\| \geq R$ for some $t_0 \in [a, \sigma^2(b)]$).*

Proofs The proofs follow the modification technique of Theorems 7 and 8 and so are omitted for brevity.

4. On Nonincreasing Solutions

Some results about the qualitative nature of solutions for the BVPs,

$$(28) \qquad\qquad y^{\Delta\Delta} = f(t, y^\sigma), \quad t \in [a, b],$$

$$(29) \qquad\qquad y(a) = A, \qquad y(\sigma^2(b)) = 0,$$

$$(30) \qquad\qquad y(a) = 0, \qquad y(\sigma^2(b)) = B.$$

are now proved. In particular, by strengthening inequality (8), the solutions furnished by Theorem 3 may be shown to be nondecreasing or nonincreasing in norm.

Corollary 1. *Let the conditions of Theorem 1 hold for the BVP* (28), (29) *with* (8) *strengthened to*

$$\langle u, f(t, u) \rangle > 0, \quad \text{for all } t \in [a, b] \text{ and all } u \neq 0. \tag{31}$$

Then the solutions to (28), (29) *guaranteed by Theorem 3 satisfy* $\|y(t)\|$ *is nonincreasing on* $[a, \sigma^2(b)]$.

Proof Note that (31) implies that $r(t) := \|y(t)\|^2$ cannot have a nonnegative maximum in $(a, \sigma^2(b))$ for any solution y, and therefore r must have a maximum at either $t = a$ or $t = \sigma^2(b)$ with $\max r(t) = \max\{r(a), r(\sigma^2(b))\} = A^2$.

Corollary 2. *Let the conditions of Theorem 3 hold for the BVP* (28), (30) *with* (8) *strengthened to*

$$\langle u, f(t, u) \rangle > 0, \quad \text{for all } t \in [a, b] \text{ and all } u \neq 0.$$

Then the solutions y to (28), (30) *guaranteed by Theorem 3 satisfy* $\|y(t)\|$ *is nondecreasing on* $[a, \sigma^2(b)]$.

Proof The proof is similar to that of Corollary 1.

5. BVPs on Infinite Intervals

This section formulates existence theorems for solutions to the following BVPs on infinite intervals:

$$y^{\Delta\Delta} = f(t, y^\sigma), \quad t \in [a, \infty), \tag{32}$$

$$y(a) = A, \quad y(t) \text{ is bounded for } t \in [a, \infty), \tag{33}$$

$$\alpha y(a) - \beta y^\Delta(a) = C, \quad y(t) \text{ is bounded for } t \in [a, \infty). \tag{34}$$

In particular, Theorems 3 and 5 will be useful.

Let $[a, \infty) = \cup_{k=1}^{\infty}[a, t_k]$. Throughout this section assume that there exists $t_n \in \mathbb{T}$ and $n \in \mathbb{N}$ such that

$$a < t_1 < t_2 < \cdots < t_n < \cdots \quad \text{with } t_n \uparrow \infty \text{ as } n \to \infty.$$

Theorem 11. *Suppose that f is continuous on $[a, \infty) \times \mathbb{R}^d$ and satisfies*

$$(35) \qquad \langle u, f(t, u) \rangle > 0, \quad \text{for all } t \in [a, \infty) \text{ and } \|u\| \geq R,$$

where $R > 0$. Then for each $\|A\| < R$, the BVP (32), (33) has at least one solution $y \in C([a, \infty); \mathbb{R}^d)$ with $\|y(t)\| < R$ on $[a, \infty)$.

Proof Fix $n \in \mathbb{N}$ and consider the BVP

$$(36) \qquad\qquad y^{\Delta\Delta} \;=\; f(t, y^{\sigma}), \quad t \in [a, t_n],$$

$$(37) \qquad\qquad y(a) = A, \qquad y(\sigma^2(t_n)) = 0.$$

It is clear from Theorem 3 that (36), (37) has a solution $y_n \in C([a, \sigma^2(t_n)]; \mathbb{R}^d)$ with $\|y_n(t)\| < R$ for $t \in [a, t_n]$. (Note also that $y_n^{\Delta\Delta} \in C_{rd}([a, \sigma^2(t_n)]; \mathbb{R}^d)$. This argument can be used for each $n \in \mathbb{N}$. The theorem then follows from Ascoli's selection theorem ([11]) applied to a sequence of intervals $[a, t_n]$ as $n \to \infty$.

Theorem 12. *Suppose that f is continuous on $[a, \infty) \times \mathbb{R}^d$ and satisfies (35), where $R > 0$. If $\alpha, \beta > 0$, then the Sturm-Liouville BVP (32), (34) has at least one solution $y \in C([a, \sigma^2(b)]; \mathbb{R}^d)$ satisfying*

$$\|y(t)\| < \max\{\|C\|/\alpha, \ R\} + 1, \quad \text{for } t \in [a, \infty).$$

Proof Fix $n \in \mathbb{N}$ and consider the BVP

$$(38) \qquad\qquad y^{\Delta\Delta} \;=\; f(t, y^{\sigma}), \quad t \in [a, t_n],$$

$$(39) \qquad\qquad \alpha y(a) - \beta y^{\Delta}(a) = C, \qquad y(\sigma^2(t_n)) = 0.$$

It is clear from Theorem 5 that (38), (39) has a solution $y_n \in C([a, \sigma^2(t_n)]; \mathbb{R}^d)$ with $\|y_n\| < M + 1$ for $t \in [a, t_n]$. (Note also that $y_n^{\Delta\Delta} \in C_{rd}([a, \sigma^2(t_n)]; \mathbb{R}^d)$. This argument can be used for each $n \in \mathbb{N}$. The theorem then follows from Ascoli's selection theorem applied to a sequence of intervals $[a, t_n]$ as $n \to \infty$.

6. On Uniqueness of Solutions

This section provides some results which guarantee the uniqueness of solutions to the Dirichlet BVP (6), (2).

Theorem 13. *If f satisfies*

$$(40) \qquad \langle u - v, f(t, u) - f(t, v) \rangle > 0, \text{ for all } t \in [a, b], \text{ and } u \neq v,$$

then (6) has, at most, one solution satisfying (2).

Proof Assume y and z are solutions to the Dirichlet BVP (6), (2). Then $y - z$ satisfies the BVP

$$y^{\Delta\Delta}(t) - z^{\Delta\Delta}(t) \;\; = \;\; f(t, y(\sigma(t))) - f(t, z(\sigma(t))), \quad t \in [a, b],$$

$$y(a) - z(a) = 0, \qquad y(\sigma^2(b)) - z(\sigma^2(b)) = 0.$$

Consider $r(t) := \|y(t) - z(t)\|^2$, $t \in [a, \sigma^2(b)]$. Now r must have a positive maximum at some point $c \in [a, \sigma^2(b)]$. From the boundary conditions, $c \in (a, \sigma^2(b))$. Choosing c as in the same fashion as in the proof of Lemma 2, it can be shown via the same reasoning that c cannot be simultaneously left-dense and right-scattered. Therefore by Lemma 1 we must have

$$(41) \qquad\qquad\qquad\qquad r^{\Delta\Delta}(\rho(c)) \leq 0.$$

So using the product rule we have

$$r^{\Delta\Delta}(\rho(c)) \geq 2\langle y^\sigma(\rho(c)) - z^\sigma(\rho(c)), f(\rho(c), y^\sigma(\rho(c))) - f(\rho(c), z^\sigma(\rho(c))) \rangle > 0,$$

which contradicts (41). (Notice at c that $\|y^\sigma(\rho(c)) - z^\sigma(\rho(c))\| = \|y(c) - z(c)\|$, since c is not simultaneously left-dense and right-scattered.) Therefore $r(t) = \|y(t) - z(t)\|^2 = 0$ for $t \in [a, \sigma^2(b)]$, and solutions of the BVP (6), (2) must be unique. This concludes the proof.

7. Uniqueness Implies Existence

In this section a uniqueness-implies-existence result is formulated for the BVP (7), (2). Since the nonlinear alternative is not required, the continuity requirements of the matrices $P(t)$ and $Q(t)$ may be relaxed to P, $Q \in C_{rd}$.

The following is a vector analogue of a result of Bohner and Peterson [6].

Theorem 14. *Let P, $Q \in C_{rd}$ and suppose the BVP*

$$(42) \qquad y^{\Delta\Delta}(t) \; = \; P(t)y^{\Delta}(\sigma(t)) + Q(t)y(\sigma(t)), \quad t \in [a, b],$$

$$(43) \qquad y(a) = 0, \qquad y(\sigma^2(b)) = 0,$$

has only the zero solution. Then the BVP (7), (2) has a unique solution for each $h \in C_{rd}([a, b]; \mathbb{R}^d)$.

Proof The proof is omitted as it follows similar lines to that of Bohner and Peterson [6] with only minor cosmetic changes.

Theorem 15. *Let $P(t)$ and $Q(t)$ be $d \times d$ matrices satisfying*

$$(44) \qquad \langle (2Q(t) - P(t)P^T(t))u, u \rangle > 0,$$

for $t \in [a, b]$, $u \neq 0$. Then (7) has a unique solution satisfying the boundary conditions (2).

Proof Since (7) is linear, the difference of two solutions to the BVP (7), (2) is also a solution of the BVP

$$(45) \qquad y^{\Delta\Delta}(t) = Q(t)y^{\sigma} + P(t)y^{\Delta}(\sigma(t)), \quad t \in [a, b]$$

$$(46) \qquad y(a) = 0, \quad y(\sigma^2(b)) = 0,$$

and it needs to be shown that the only solution to (45), (46) is $y = 0$.

Assume the contrary, let y be a nontrivial solution to (45), (46) and put $r(t) = \|y(t)\|^2$. Now r must have a positive maximum at some point $c \in [a, \sigma^2(b)]$. ¿From the boundary conditions, $c \in (a, \sigma^2(b))$. Choosing c as in the same fashion as in the proof

of Lemma 2 it can be shown via the same reasoning that c cannot be simultaneously left-dense and right-scattered. Therefore by Lemma 1

$$(47) \qquad\qquad r^{\Delta\Delta}(\rho(c)) \leq 0.$$

So using the product rule we have

$$r^{\Delta\Delta}(\rho(c)) = 2\langle y^{\sigma}(\rho(c)), f(\rho(c), y^{\sigma}(\rho(c)))\rangle + \|y^{\Delta}(\rho(c))\|^2 + \|[y^{\sigma\Delta}(\rho(c))]\|^2.$$

Using the identity $\langle Ab, c\rangle = \langle b, A^T c\rangle$, it can be verified that

$$2\langle P(\rho(c))y^{\Delta}(\sigma(\rho(c))) + Q(\rho(c))y^{\sigma}(\rho(c)), y^{\sigma}(\rho(c))\rangle + \|y^{\Delta}(\sigma(\rho(c)))\|^2$$
$$= \|y^{\Delta}(\sigma(\rho(c))) + P^T(\rho(c))y^{\sigma}(\rho(c))\|^2$$
$$+ \langle (2Q(\rho(c)) - P(\rho(c))P^T(\rho(c)))y^{\sigma}(\rho(c)), y^{\sigma}(\rho(c))\rangle,$$

and therefore

$$r^{\Delta\Delta}(\rho(c)) \geq \langle (2Q(\rho(c)) - P(\rho(c))P^T(\rho(c)))y^{\sigma}(\rho(c)), y^{\sigma}(\rho(c))\rangle.$$

Hence (44) implies $r^{\Delta\Delta}(\rho(c)) > 0$ which contradicts (47). It follows that $r(t) = 0$ on $[a, \sigma^2(b)]$. That is, the only solution to (45), (46) is $y = 0$. The existence of solutions for (7), (2) now follows from Theorem 14 and this completes the proof.

The paper is now concluded with an example of a nonlinear vector BVP on a number of different time scales. First we give $y^{\Delta\Delta}$ for three simple examples.

Example Let $\mathbb{T} = \mathbb{Z}$. Here $\sigma(t) = t + 1$ and

$$y^{\Delta\Delta}(t) = y(t+2) - 2y(t+1) + y(t).$$

Example Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$. Here $\sigma(t) = t + h$ and

$$y^{\Delta\Delta}(t) = \frac{y(t+2h) - 2y(t+h) + y(t)}{h^2}.$$

Example Let $q > 1$ and $\mathbb{T} = q^{\mathbb{N}_0}$ Here $\sigma(t) = qt$ and

$$y^{\Delta\Delta}(t) = \frac{y(q^2 t) - (q+1)y(qt) + qy(t)}{q(q-1)^2 t^2}.$$

Dynamic equations on the time scale $\mathbb{T} = q^{\mathbb{N}_0}$ are called q-difference equations and they have many important applications of q-difference equations.

Example Consider the BVP

$$
(48) \qquad \begin{pmatrix} y_1^{\Delta\Delta} \\ y_2^{\Delta\Delta} \end{pmatrix} = \begin{pmatrix} t^2(y_1^\sigma)^3 - y_2^\sigma \\ y_1^\sigma + y_2^\sigma \exp(y_1^\sigma y_2^\sigma) \end{pmatrix}, \quad t \in [a,b], \ a > 0
$$

subject to the boundary conditions

$$
(49) \qquad (y_1(a), y_2(a)) = (1,1), \quad (y_1(\sigma^2(b)), y_2(\sigma^2(b))) = (0,0).
$$

Note that the conditions of Theorem 5 are satisfied, and thus the BVP (48), (49) has a solution $y = (y_1, y_2)$ on each of the above time scales, $\mathbb{T} = \mathbb{Z}$, $h\mathbb{Z}$ and $q^{\mathbb{N}_0}$, satisfying $\|y(t)\| < 2$. Note also that the inequality (31) holds, and therefore $\|y(t)\|$ is nonincreasing on $[a, \sigma^2(b)]$.

Remark 2. *The question on how to ensure $[y(\sigma(t)]^\Delta$ exists is naturally raised. This "smoothness" requirement will be satisfied if, for example, σ is differentiable or when the points in $\mathbb{T}$ are isolated (left-scattered and right-scattered). However, if these cases are excluded then the method of upper and lower solutions for systems of equations can be developed with the a priori bounds on solutions being obtained in a component-wise fashion. The arguments are essentially minor extensions of the ideas in [4].*

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