Comparison Theorems for Linear Dynamic Equations on Time Scales

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ABSTRACT: We obtain several comparison theorems for second order linear dynamic equations on a time scale. These results extend comparison theorems for the continuous case and provide some new results in the discrete case, as well as other more general situations.

AMS Subject Classification. 34B10, 39A10.

1. INTRODUCTION

Let $\mathbb{T}$ be a time scale (i.e., a closed subset of $\mathbb{R}$) with $\sup \mathbb{T} = \infty$. We shall be interested in obtaining comparison theorems for the second order linear equations

$$[p(t)x^\Delta(t)]^\Delta + q(t)x^\sigma(t) = 0, \tag{1}$$
$$[p(t)y^\Delta(t)]^\Delta + a^\sigma(t)q(t)y^\sigma(t) = 0, \tag{2}$$
$$[p(t)z^\Delta(t)]^\Delta + a(t)q(t)z^\sigma(t) = 0, \tag{3}$$

where $p(t) > 0$ and p, q, a are right-dense continuous on $\mathbb{T}$.
For completeness, we introduce the following concepts related to the notion of time scales. We refer to [1] for additional details concerning the calculus on time scales.

[1]Research was supported by NSF Grant 0072505
[2]Supported by Grants 201/01/P041 and 201/01/0079 of the Czech Grant Agency; research conducted while visiting the University of Nebraska-Lincoln

Definition 1. Let $\mathbb{T}$ be a time scale and define the *forward jump operator* $\sigma(t)$ at t , for $t \in \mathbb{T}$, by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the *backward jump* operator $\rho(t)$ at t, for $t \in \mathbb{T}$, by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. If $\sigma(t) = t$ we say t is *right-dense*, while if $\rho(t) = t$ we say t is *left-dense*. Given an interval $[c, d]$ in $\mathbb{T}$ the notation $[c, d]^\kappa$ denotes the interval $[c, d]$ in case $\rho(d) = d$ and denotes the interval $[c, d)$ in case $\rho(d) < d$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous provided f is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. We shall also use the notation $\mu(t) := \sigma(t) - t$ which is called the graininess function.

Definition 2. We say that a solution x of (1) has a *generalized zero* at t in case $x(t) = 0$. We say x has a *generalized zero* in $(t, \sigma(t))$ in case $x(t)x(\sigma(t)) < 0$ and $\mu(t) > 0$. We say that (1) is *disconjugate* on the interval $[c, d]$, if there is no nontrivial solution of (1) with two (or more) generalized zeros in $[c, d]$.

Definition 3. Equation (1) is said to be *nonoscillatory* on $[\tau, \infty)$ if there exists $c \in [\tau, \infty)$ such that this equation is disconjugate on $[c, d]$ for every $d > c$. In the opposite case (1) is said to be *oscillatory* on $[\tau, \infty)$. Oscillation of (1) may equivalently be defined as follows. A nontrivial solution y of (1) is called *oscillatory* if it has infinitely many (isolated) generalized zeros in $[\tau, \infty)$. By the Sturm type separation theorem, one solution of (1) is (non)oscillatory iff every solution of (1) is (non)oscillatory. Hence we can speak about *oscillation* or *nonoscillation of equation* (1).

Basic oscillatory properties of (1) are described by the so-called Reid Roundabout Theorem which is proved e.g. in [1, Theorem 4.12, Theorem 4.53, Theorem 4.57].

Proposition 1 (Reid Roundabout Theorem). *The following statements are equivalent:*

(i) Equation (1) *is disconjugate on* $[c, d]$*.*

(ii) Equation (1) *has a solution without generalized zeros on* $[c, d]$*.*

(iii) The Riccati dynamic equation

$$u^{\Delta}(t) + q(t) + \frac{u^2(t)}{p(t) + \mu(t)u(t)} = 0 \tag{4}$$

has a solution u *with* $p(t) + \mu(t)u(t) > 0$ *for* $t \in [c, d]^{\kappa}$ *(except for the left-dense right-scattered* d *at which* $p + \mu u$ *may be nonpositive).*

(iv) The quadratic functional

$$\mathcal{F}(\xi; c, d) = \int_c^d \left\{ p(t) \left(\xi^{\Delta}(t)\right)^2 - q(t)(\xi^{\sigma}(t))^2 \right\} \Delta t$$

is positive definite for $\xi \in U(c, d)$*, where*

$$U(c, d) = \{\xi \in C_p^1[c, d] \, : \, \xi(c) = \xi(d) = 0\}.$$

This proposition makes it therefore clear that there are at least two methods of investigation of (non)oscillation of (1). The first one – the *variational method* – is based on the equivalence of (i) and (iv) and its basic statement can be reformulated as follows:

Lemma 4 (Variational method). *If for any* $T \in [\tau, \infty)$ *there exists* $0 \not\equiv \xi \in U(T)$*, where*

$$U(T) = \{\xi \in C_p^1[T, \infty) \, : \, \xi(t) = 0 \text{ for } t \in [\tau, T] \text{ and } \exists \hat{T}, \hat{T} > \sigma(T), \\ \text{such that } \xi(t) = 0 \text{ for } t \in [\sigma(\hat{T}), \infty)\},$$

such that $\mathcal{F}(\xi; T, \infty) = \mathcal{F}(\xi, T, \sigma(\hat{T})) \leq 0$*, then* (1) *is oscillatory.*

Another method of investigation for the oscillation theory of (1) is based on the equivalence of (i) and (iii) in Proposition 1. This is usually referred to as the *Riccati technique* and by virtue of the Sturm Comparison Theorem implies that for nonoscillation of (1), it is sufficient to find a solution of the Riccati-type inequality as given in the next lemma. A proof may be found in [3] or [1].

Lemma 5 (Riccati technique). *Equation* (1) *is nonoscillatory if and only if there exists* $T \in [\tau, \infty)$ *and a function* u *satisfying the Riccati dynamic inequality*

$$u^{\Delta}(t) + q(t) + \frac{u^2(t)}{p(t) + \mu(t)u(t)} \leq 0$$

with $p(t) + \mu(t)u(t) > 0$ *for* $t \in [T, \infty)$.

For completeness, we recall the following

Lemma 6 (Sturm-Picone Comparison Theorem). *Consider the equation*

$$[\tilde{p}(t)x^{\Delta}(t)]^{\Delta} + \tilde{q}(t)x^{\sigma}(t) = 0, \tag{5}$$

where $\tilde{p}$ *and* $\tilde{q}$ *satisfy the same assumptions as* p *and* q. *Suppose that* $\tilde{p}(t) \leq p(t)$ *and* $q(t) \leq \tilde{q}(t)$ *on* $[T, \infty)$ *for all large* T. *Then* (5) *is nonoscillatory on* $[\tau, \infty)$ *implies* (1) *is nonoscillatory on* $[\tau, \infty)$.

2. MAIN RESULTS

We mention first a few background details which serve to motivate the results in this paper. Suppose that $\mathbb{T}$ is the real interval $[0, +\infty)$ so that (1) becomes

$$[p(t)x'(t)]' + q(t)x(t) = 0, \tag{6}$$

where $p(t)$ is continuous and positive and $q(t)$ is continuous on $[0, +\infty)$. It was shown in [2] that if (6) is oscillatory, then multiplying the coefficient $q(t)$ by a function $a(t)$ where $a(t) \geq 1$ and $p(t)a'(t)$ is nonincreasing preserves oscillation; i.e.,

$$[p(t)x'(t)]' + a(t)q(t)x(t) = 0, \tag{7}$$

is also oscillatory. Of course, if $q(t)$ is nonnegative, these results follow immediately from the usual Sturm-Picone Comparison Theorem, but when $q(t)$ changes sign on each half line, oscillation of (7) is not obvious if (6) is oscillatory. One may also notice that if (6) is oscillatory and if $a(t) = const = \lambda \geq 1$ then oscillation of (7) follows immediately from the Sturm-Picone Theorem by dividing the equation by λ (for the case when $q(t)$ may change sign). This result, i.e., the statement that says that if (6) is oscillatory, then so is

$$(p(t)y')' + \lambda q(t)y = 0$$

for any constant $\lambda \geq 1$, was also observed by Fink and St. Mary [5]. Kwong in [8] then showed that the result of [2] may be strengthened to a larger class of functions $a(t)$ by relaxing somewhat the monotonicity assumption on $p(t)a'(t)$ as given in [2]. We present below three different comparison theorems along with their corresponding corollaries, and show by examples, that they are all independent. In addition to extending the results of [8] and [2] in the case of equations (6) and (7) in the continuous case, the results we obtain are new in the discrete case and the more general time scales case. It should also be noted that because of the techniques of proof used, both (2) and (3) may be viewed as the time-scales extensions of (1), obtained when multiplying $q(t)$ by $a(t)$ (which is the same as $a^\sigma(t)$ when $\mathbb{T} = \mathbb{R}$.) We shall state the results in this section but defer the proofs and examples to the following two sections.

Our first result shows that if, "on average", $q(t)$ is more positive than negative, then the assumptions on $a(t)$ are quite mild. To be precise, we have

Theorem 7. *Assume $a \in C^1_{rd}$ and*

(i) $\liminf_{t\to\infty} \int_T^t q(s)\Delta s \geq 0$ *but* $\not\equiv 0$ *for all large* T,

(ii) $\int_\tau^\infty \frac{1}{p(s)}\Delta s = \infty$,

(iii) $0 < a(t) \leq 1, \quad a^\Delta(t) \leq 0.$

Then (1) *is nonoscillatory on* $[\tau, \infty)$ *implies* (3) *is nonoscillatory on* $[\tau, \infty)$.

The corresponding "oscillation" result is

Corollary 8. *Assume $a \in C^1_{rd}$ and*

(i) $\liminf_{t\to\infty} \int_T^t a(s)q(s)\Delta s \geq 0$ *but* $\not\equiv 0$ *for all large* T,

(ii) $\int_\tau^\infty \frac{1}{p(s)}\Delta s = \infty$,

(iii) $a(t) \geq 1, \quad a^\Delta(t) \geq 0.$

Then (1) *is oscillatory on* $[\tau, \infty)$ *implies* (3) *is oscillatory on* $[\tau, \infty)$.

If we strengthen the assumptions on $a(t)$ somewhat, then we may relax the assumptions on $q(t)$ and in this case, we consider the relation between (1) and (2). For convenience, we state first the "oscillation" result.

Theorem 9. *Assume $pa^\Delta \in C^1_{rd}$ and*

(i) $a(t) \geq 1$,

(ii) $\mu(t)a^\Delta(t) \geq 0$,

(iii) $\left(p(t)a^\Delta(t)\right)^\Delta \leq 0$.

Then (1) *is oscillatory on* $[\tau, \infty)$ *implies* (2) *is oscillatory on* $[\tau, \infty)$

In this case, the analogous "nonoscillation" result becomes

Corollary 10. *If* $p(\frac{1}{a})^\Delta \in C^1_{rd}$,

(i) $0 < a(t) \leq 1$,

(ii) $\mu(t)a^\Delta(t) \leq 0$,

(iii) $\left(p(t)\left(\frac{1}{a(t)}\right)^\Delta\right)^\Delta \leq 0$.

Then (1) *is nonoscillatory on* $[\tau, \infty)$ *implies* (2) *is nonoscillatory on* $[\tau, \infty)$

In the following theorem we let χ denote the characteristic function of the set of right-scattered points $\hat{\mathbb{T}}$ defined by

$$\hat{\mathbb{T}} := \{t \in \mathbb{T} : \mu(t) > 0\}.$$

That is,

$$\chi(t) := \begin{cases} 1, & t \in \hat{\mathbb{T}} \\ 0, & t \notin \hat{\mathbb{T}}. \end{cases}$$

Theorem 11. *Assume $pa^\Delta \in C^1_{rd}$, and that the following conditions hold:*

(i) $a(t) > 0$ *and* $2a(t) + \mu(t)a^\Delta(t) \leq 2$

(ii) $p(t) > \epsilon_1\mu(t)$ *for some* $\epsilon_1 > 0$ *and for all* $t \in \mathbb{T}$,

(iii) there is an $\epsilon_0 > 0$ *such that the function*

$$G_{\epsilon_0}(t) := 2\left(a^\Delta(t)p(t)\right)^\Delta - \frac{\left(a^\Delta(t)p(t)\right)^2}{a(t)\left(p(t) - \mu(t)\epsilon_0\right)} \geq 0$$

for all large t, *where* $p(t) - \mu(t)\epsilon_0 > 0$, *(iv)* $\limsup_{t\to\infty} \chi(t) \int_\tau^t q(s)\Delta s > -\infty$,

(v) there exists a constant $M > 0$ *such that* $\chi(t)p(t) \le M\mu(t)$, *for* $t \in \mathbb{T}$.

Then (1) *is nonoscillatory on* $[\tau, \infty)$ *implies* (2) *is nonoscillatory on* $[\tau, \infty)$

Again, we have a corresponding "oscillation" result:

Corollary 12. *Assume* $p\left(\frac{1}{a}\right)^\Delta \in C^1_{rd}$ *and that the following conditions hold:*

(i) $a(t) + \frac{\mu(t)a^\Delta(t)}{2a^\sigma(t)} \ge 1$ *for all large* t,

(ii) there is an $\epsilon_1 > 0$ *such that* $p(t) > \mu(t)\epsilon_1$ *for* $t \in \mathbb{T}$,

(iii) there exists $\epsilon_0 > 0$ *such that the function*

$$H_{\epsilon_0}(t) := 2\left(\delta(t)\right)^\Delta + \frac{a(t)\delta^2(t)}{p(t) - \mu(t)\epsilon_0} \le 0,$$

for all large t, *where* $p(t) - \mu(t)\epsilon_0 > 0$, *and where*

$$\delta(t) := \frac{p(t)a^\Delta(t)}{a(t)a^\sigma(t)} = -p(t)\left(\frac{1}{a(t)}\right)^\Delta,$$

(iv) $\limsup_{t\to\infty} \chi(t) \int_\tau^t a^\sigma(s)q(s)\Delta s > -\infty$,

(v) there is an $M > 0$ *such that* $\chi(t)p(t) \le M\mu(t)$ *for all large* t.

Then (1) *is oscillatory on* $[\tau, \infty)$, *implies that* (2) *is oscillatory on* $[\tau, \infty)$.

We notice in the last two results how the graininess function is involved in the criteria for oscillation/nonoscillation. In particular, for the case when $\mathbb{T} = [0, +\infty)$, then $\mu(t) \equiv 0$, so conditions (ii), (iv), and (v) of Corollary 12 hold trivially, and it may be shown that (iii) reduces to the condition of Kwong in [8]. The nonoscillation result Theorem 11 is new in all cases.

3. PROOFS

We begin this section with an auxiliary statement showing that under certain assumptions, any positive solution of (1) must be eventually increasing.

Lemma 13. *Assume*

$$\liminf_{t\to\infty} \int_T^t q(s)\Delta s \geq 0 \quad \textit{and} \quad \not\equiv 0 \tag{8}$$

for all large T, and

$$\int_\tau^\infty \frac{1}{p(s)}\Delta s = \infty. \tag{9}$$

If x is a solution of (1) *such that $x(t) > 0$ for $t \in [T, \infty)$, then there exists $S \in [T, \infty)$ such that $x^\Delta(t) > 0$ for $t \in [S, \infty)$.*

Proof. The proof is by contradiction. We consider two cases:

(i) Suppose that $x^\Delta(t) < 0$ for $t \in [T, \infty)$. Define $Q(t, T) = \int_T^t q(s)\Delta s$. We may assume, by condition (8), that T is such that $Q(t, T) \geq 0$, $t \in [T, \infty)$. Indeed, if no such T exists, then for $T \in [\tau, \infty)$ fixed but arbitrary, we define

$$T_1 = T_1(T) := \sup\left\{ t > T \, : \, \int_T^t q(s)\Delta s < 0 \right\}.$$

If $T_1 = \infty$, then choosing $t_n \to \infty$ such that $Q(t_n, T) < 0$ for all n. we obtain a contradiction to (8). Hence, we must have $T_1 < \infty$ which implies $Q(t, T_1) \geq 0$ for $t \in [T_1, \infty)$. Now an integration by parts gives (with $T_1 = T$)

$$\begin{aligned} \int_T^t q(s)x^\sigma(s)\Delta s &= \int_T^t Q^\Delta(s, T)x^\sigma(s)\Delta s \\ &= Q(t, T)x(t) - \int_T^t Q(s, T)x^\Delta(s)\Delta s \geq 0. \end{aligned}$$

Integrating (1) we have, from this last estimate,

$$x^\Delta(t) \leq \frac{p(T)x^\Delta(T)}{p(t)} \tag{10}$$

for $t \in [T, \infty)$. Integrating (10) for $t \geq T$ we see that $x(t) \to -\infty$ by (9), a contradiction. Therefore, $x^\Delta(t) < 0$ cannot hold for all large t.

(ii) Next, if $x^\Delta(t) \not> 0$ eventually, then for every (large) $T \in [\tau, \infty)$ there exists $T_0 \in [T, \infty)$ such that $x^\Delta(T_0) \leq 0$ and we may suppose that $\liminf_{t\to\infty} \int_{T_0}^t q(s)\Delta s \geq 0$. Since $x(t) > 0$ for $t \in [T, \infty)$, the function $u(t) :=$

$p(t)x^\Delta(t)/x(t)$ satisfies the Riccati equation (4) with $p(t)+\mu(t)u(t)>0$ for $t\in[T,\infty)$. Integrating (4) from T_0 to t, $t\geq T_0$, gives

$$u(t)=u(T_0)-\int_{T_0}^t q(s)\Delta s-\int_{T_0}^t \frac{u^2(s)}{p(s)+\mu(s)u(s)}\Delta s.$$

Therefore, it follows that $\limsup_{t\to\infty} u(t)<0$, using the facts that $u(T_0)\leq 0$, $u(t)$ is eventually nontrivial, and (8) holds. Hence there exists $T_2\in[T,\infty)$ such that $u(t)<0$ for $t\in[T_2,\infty)$ and so $x^\Delta(t)<0$ for $t\in[T_2,\infty)$, a contradiction to the first part. □

Proof of Theorem 7:

Proof. The assumptions of the theorem imply that there exists a solution x of (1) and $T\in[\tau,\infty)$ such that $x(t)>0$ and $x^\Delta(t)>0$ on $[T,\infty)$ by Lemma 13. Therefore, the function $u(t):=p(t)x^\Delta(t)/x(t)>0$ satisfies (4) with $p(t)+\mu(t)u(t)>0$ on $[T,\infty)$. Multiplying (4) by $a(t)$ we get

$$\begin{aligned}0 &= u^\Delta(t)a(t)+a(t)q(t)+\frac{a(t)u^2(t)}{p(t)+\mu(t)u(t)}\\ &\geq u^\Delta(t)a(t)+u^\sigma(t)a^\Delta(t)+a(t)q(t)+\frac{[a(t)u(t)]^2}{a(t)p(t)+\mu(t)a(t)u(t)}\\ &= [a(t)u(t)]^\Delta+a(t)q(t)+\frac{[a(t)u(t)]^2}{a(t)p(t)+\mu(t)a(t)u(t)}\end{aligned}$$

for $t\in[T,\infty)$. Hence the function $v(t)=a(t)u(t)$ satisfies the Riccati inequality

$$v^\Delta(t)+a(t)q(t)+\frac{v^2(t)}{a(t)p(t)+\mu(t)v(t)}\leq 0$$

with $a(t)p(t)+\mu(t)v(t)>0$ for $t\in[T,\infty)$. Therefore, the equation

$$[a(t)p(t)z^\Delta(t)]^\Delta+a(t)q(t)z^\sigma(t)=0 \tag{11}$$

is nonoscillatory by Lemma 5 and so equation (3) is nonoscillatory by Lemma 6 since $a(t)p(t)\leq p(t)$. □

Proof of Corollary 8:

Proof. The proof is by contradiction and is similar to that of Theorem 7 where we divide, instead of multiply, the corresponding Riccati equation by the function $a(t)$. □

Proof of Theorem 9:

Proof. Oscillation of (1) implies that for every $T \in [\tau, \infty)$ there exist $c, d \in \mathbb{T}$, $T \leq c \leq \sigma(c) < d$, and a solution x of (1) such that

$$x(c) = 0 \text{ if } \mu(c) = 0,\ \ x(c)x^\sigma(c) \leq 0 \text{ if } \mu(c) > 0,$$
$$x(d) = 0 \text{ if } \mu(d) = 0,\ \ x(d)x^\sigma(d) \leq 0 \text{ if } \mu(d) > 0,$$

and $x(t) \neq 0$ in (c, d). Define

$$\xi(t) := \begin{cases} 0, & \tau \leq t \leq c, \\ x(t), & c < t \leq d, \\ 0, & d < t. \end{cases}$$

We show that

$$\mathcal{F}_a(\xi) = \int_T^{\sigma(S)} \left\{ a(t)p(t)\left(\xi^\Delta(t)\right)^2 - a^\sigma(t)q(t)(\xi^\sigma(t))^2 \right\} \Delta t \leq 0.$$

Using integration by parts we have

$$\begin{aligned}
\mathcal{F}_a(\xi) &= \int_c^{\sigma(c)} a(t)p(t)(\xi^\Delta(t))^2\Delta t + \int_{\sigma(c)}^d a(t)p(t)(x^\Delta(t))^2\Delta t + \\
&\quad + \int_d^{\sigma(d)} a(t)p(t)(\xi^\Delta(t))^2\Delta t - \int_c^{\sigma(c)} a^\sigma(t)q(t)(\xi^\sigma(t))^2\Delta t - \\
&\quad - \int_{\sigma(c)}^d a^\sigma(t)q(t)(x^\sigma(t))^2\Delta t - \int_d^{\sigma(d)} a^\sigma(t)q(t)(\xi^\sigma(t))^2\Delta t \\
&= \left[a(t)p(t)\xi^\Delta(t)\xi(t)\right]_c^{\sigma(c)} - \int_c^{\sigma(c)} (p(t)\xi^\Delta(t))^\Delta a^\sigma(t)\xi^\sigma(t)\Delta t - \\
&\quad - \int_c^{\sigma(c)} p(t)\xi^\Delta(t)a^\Delta(t)\xi^\sigma(t)\Delta t + \left[a(t)p(t)x^\Delta(t)x(t)\right]_{\sigma(c)}^d - \\
&\quad - \int_{\sigma(c)}^d (p(t)x^\Delta(t))^\Delta a^\sigma(t)x^\sigma(t)\Delta t - \int_{\sigma(c)}^d p(t)\xi^\Delta(t)a^\Delta(t)\xi^\sigma(t)\Delta t + \\
&\quad + \mu(d)a(d)p(d)(\xi^\Delta(d))^2 - \mu(c)a^\sigma(c)q(c)(\xi^\sigma(c))^2 - \\
&\quad - \int_{\sigma(c)}^d a^\sigma(t)q(t)(x^\sigma(t))^2\Delta t - \mu(d)a^\sigma(d)q(d)(\xi^\sigma(d))^2 \\
&= A + C + D - \int_{\sigma(c)}^d \left[(p(t)x^\Delta(t))^\Delta + q(t)x^\sigma(t)\right] x^\sigma(t)a^\sigma(t)\Delta t,
\end{aligned}$$

where

$$A = -\int_c^d p(t)\xi^\Delta(t)a^\Delta(t)\xi^\sigma(t)\Delta t,$$

$$C = -\mu(c)a^\sigma(c)x^\sigma(c)\left[(p(c)\xi^\Delta(c))^\Delta + q(c)x^\sigma(c)\right]$$

and

$$D = a(d)p(d)\left[x^\Delta(d)x(d) + \mu(d)(\xi^\Delta(d))^2\right].$$

We claim that $A \le 0$. If

$$\int_c^d p(t)\xi^\Delta(t)a^\Delta(t)\xi(t)\Delta t \le 0,$$

then we have

$$\begin{aligned}
A &\le -\int_c^d p(t)\xi^\Delta(t)a^\Delta(t)(\xi^\sigma(t)+\xi(t))\Delta t = \int_c^d p(t)(\xi^2(t))^\Delta a^\Delta(t)\Delta t \\
&= \left[p(t)\xi^2(t)a^\Delta(t)\right]_c^d + \int_c^d (p(t)a^\Delta(t))^\Delta(\xi^\sigma(t))^2\Delta t \\
&\le -p(d)(\xi^2(d))a^\Delta(d) \le 0,
\end{aligned}$$

since we have $a^\Delta(d) \ge 0$ for $\mu(d) > 0$ and $\xi(d) = x(d) = 0$ for $\mu(d) = 0$. On the other hand, if

$$\int_c^d p(t)\xi^\Delta(t)a^\Delta(t)\xi(t)\Delta t \ge 0,$$

then

$$\begin{aligned}
A &\le -\int_c^d p(t)\xi^\Delta(t)a^\Delta(t)(\xi^\sigma(t)-\xi(t))\Delta t \\
&= -\int_c^d p(t)(\xi^\Delta(t))^2\mu(t)a^\Delta(t)\Delta t \le 0.
\end{aligned}$$

Clearly $C = 0$ for $\mu(c) = 0$ and $D = 0$ for $\mu(d) = 0$. If $\mu(c) > 0$, then

$$\begin{aligned}
C &= -\mu(c)a^\sigma(c)x^\sigma(c)\left\{\frac{p(\sigma(c))x^\Delta(\sigma(c)) - p(c)\xi^\Delta(c)}{\mu(c)} + q(c)x^\sigma(c)\right\} \\
&= -\mu(c)a^\sigma(c)x^\sigma(c)\Big\{(p(c)x^\Delta(c))^\Delta + \\
&\quad + \frac{p(c)}{\mu^2(c)}\left[x^\sigma(c) - x(c) - \xi^\sigma(c) + \xi(c)\right] + q(c)x^\sigma(c)\Big\} \\
&= \frac{a^\sigma(c)p(c)x(c)x^\sigma(c)}{\mu(c)} \le 0.
\end{aligned}$$

To prove that

$$D = \frac{a(d)p(d)x(d)x^\sigma(d)}{\mu(d)} \leq 0$$

we proceed in the same way as in the proof of [1, Theorem 4.56]. Therefore, we get $\mathcal{F}_a(\xi) \leq 0$ and so the equation

$$[a(t)p(t)y^\Delta(t)]^\Delta + a^\sigma(t)q(t)y^\sigma(t) = 0 \tag{12}$$

is oscillatory by Lemma 4. Hence, since $p(t) \leq a(t)p(t)$, it follows that equation (2) is oscillatory by Lemma 6. ☐

Proof of Corollary 10:

Proof. The proof is by contradiction and is based on the fact that if the equation

$$\left[p(t)y^\Delta\right]^\Delta + \frac{1}{b^\sigma(t)}q(t)y^\sigma = 0 \tag{13}$$

is oscillatory, then (1) is oscillatory by Theorem 9 provided

$$b(t) \geq 1, \quad \mu(t)b^\Delta(t) \geq 0, \quad (p(t)b^\Delta(t))^\Delta \leq 0. \tag{14}$$

Suppose, for the sake of contradiction, that (2) is oscillatory and set $b(t) = 1/a(t)$. Then (13) is equivalent to (2) and hence is oscillatory. Moreover, (14) is satisfied since we have $b(t) = 1/a(t) \geq 1$, $\mu(t)b^\Delta(t) = \mu(t)(1/a(t))^\Delta = -\mu(t)a^\Delta(t)/(a(t)a^\sigma(t)) \geq 0$ and $(p(t)b^\Delta(t))^\Delta = (p(t)(1/a(t))^\Delta)^\Delta \leq 0$ by (i),(ii) and (iii) of Corollary 10. Therefore, equation (1) is oscillatory, a contradiction. ☐

Proof of Theorem 11:

Proof. We suppose that (1) is nonoscillatory and without loss of generality, suppose that x is a solution of (1) with $x(t)x^\sigma(t) > 0$, for $t \geq T_1 \geq \tau$. Then letting

$$u(t) := \frac{p(t)x^\Delta(t)}{x(t)},$$

for $t \geq T_1$, it follows that u satisfies the Riccati equation (4) for $t \geq T_1$, and $p(t) + \mu(t)u(t) > 0$, for $t \geq T_1$. An integration of (4) for $t \geq T_1$ gives

$$u(t) + \int_{T_1}^t q(s)\Delta s + \int_{T_1}^t F(s)\Delta s = u(T_1), \tag{15}$$

where

$$F(t) := \frac{u^2(t)}{p(t) + \mu(t)u(t)}, \quad t \geq T_1.$$

Since $F(t) \geq 0$, it follows that

$$\lim_{t\to\infty} \int_{T_1}^{t} F(s)\Delta s = L \leq \infty.$$

Now we claim that if $\{t_k\} \subset \hat{\mathbb{T}}$ is such that $\lim_{k\to\infty} t_k = +\infty$ then

$$\lim_{k\to\infty} u(t_k) = 0.$$

(We remark that if such a sequence does not exist, then $[T, \infty)$ is a real interval for sufficiently large T and we may proceed in the proof by introducing the new variable R as in (17) below with $\mu(t) \equiv 0$ on $[T, \infty)$.) For the sequence t_k we note that since $p(t_k) + \mu(t_k)u(t_k) > 0$ we have

$$u(t_k) \geq -\frac{p(t_k)}{\mu(t_k)} \geq -M$$

by part (v) of the hypotheses of the theorem. Hence we have from (15),

$$-M + \int_{T_1}^{t_k} q(s)\Delta s + \int_{T_1}^{t_k} F(s)\Delta s \leq u(T_1). \tag{16}$$

Since

$$\limsup_{k\to\infty} \int_{T_1}^{t_k} q(s)\Delta s > -\infty$$

by part (iv) of the hypotheses, it follows from (16) that $\lim_{k\to\infty} \int_{T_1}^{t_k} F(s)\Delta s = L < +\infty$, which implies that

$$\lim_{t\to\infty} \int_{T_1}^{t} F(s)\Delta s = L < +\infty.$$

To see that $\lim_{k\to\infty} u(t_k) = 0$, notice that

$$0 \leq \sum_{k=1}^{\infty} \mu(t_k)F(t_k) = \sum_{k=1}^{\infty} \int_{t_k}^{\sigma(t_k)} F(s)\Delta s \leq \int_{T_1}^{\infty} F(s)\Delta s < \infty,$$

and so

$$\lim_{k\to\infty} \mu(t_k)F(t_k) = \lim_{k\to\infty} \frac{\mu(t_k)u^2(t_k)}{p(t_k) + \mu(t_k)u(t_k)} = 0.$$

Therefore given $\epsilon > 0$, choose $k_0 \geq 1$ so that

$$0 < \frac{\mu(t_k)u^2(t_k)}{p(t_k) + \mu(t_k)u(t_k)} < \epsilon,$$

for $k \geq k_0$. Thus,

$$u^2(t_k) < \epsilon \left(\frac{p(t_k)}{\mu(t_k)} + u(t_k) \right), \quad k \geq k_0.$$

Solving this quadratic inequality gives

$$\left(u(t_k) - \frac{\epsilon}{2} \right)^2 < \frac{\epsilon^2}{4} + \epsilon \frac{p(t_k)}{\mu(t_k)} \leq \frac{\epsilon^2}{4} + \epsilon M$$

and so

$$\left| u(t_k) - \frac{\epsilon}{2} \right| < \frac{\epsilon}{2} + \sqrt{\epsilon M}, \quad k \geq k_0,$$

which implies

$$|u(t_k)| < \epsilon + \sqrt{\epsilon M}, \quad k \geq k_0.$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$\lim_{k\to\infty} u(t_k) = 0.$$

Now in the Riccati equation (4) we let

$$R := au \tag{17}$$

and obtain

$$R^\Delta = a^\sigma u^\Delta + a^\Delta u$$

and so it follows that R solves the dynamic equation

$$R^\Delta - \frac{a^\Delta(t)}{a(t)} R + a^\sigma(t)q(t) + \frac{a^\sigma(t)R^2}{a^2(t)p(t) + \mu(t)a(t)R} = 0.$$

After some manipulations, we obtain (using $a^\sigma = a + \mu a^\Delta$),

$$R^\Delta + a^\sigma(t)q(t) + \frac{R^2 - a^\Delta(t)p(t)R}{a(t)p(t) + \mu(t)R} = 0$$

which shows that R solves

$$R^{\Delta}+a^{\sigma}(t)q(t)+\frac{(R-\frac{1}{2}a^{\Delta}(t)p(t))^2-\frac{1}{4}(a^{\Delta}(t)p(t))^2}{a(t)p(t)+\mu(t)R}=0 \tag{18}$$

for $t\geq T_1$. Let us now set $\hat{R}=R-\frac{1}{2}a^{\Delta}p$ so that (18) becomes

$$\hat{R}^{\Delta}(t)+a^{\sigma}(t)q(t)+\frac{\hat{R}^2(t)}{p_1(t)+\mu(t)\hat{R}(t)}+G(t)=0, \tag{19}$$

where

$$G(t)=\left(\frac{1}{2}a^{\Delta}(t)p(t)\right)^{\Delta}-\frac{1}{4}\frac{(a^{\Delta}(t)p(t))^2}{a(t)p(t)+\mu(t)R(t)}$$

and

$$p_1=ap+\frac{1}{2}\mu a^{\Delta}p.$$

We claim that $G(t)\geq 0$ for all large t, say for $t\geq T_2\geq T_1$. Assuming for the moment that this is the case, then from (19) we have that $\hat{R}$ is a solution of the Riccati dynamic inequality

$$\hat{R}^{\Delta}(t)+a^{\sigma}(t)q(t)+\frac{\hat{R}^2(t)}{p_1(t)+\mu(t)\hat{R}(t)}\leq 0, \tag{20}$$

and $p_1(t)+\mu(t)\hat{R}(t)>0$. Therefore, it follows by Lemma 5 that the equation

$$(p_1(t)y^{\Delta})^{\Delta}+a^{\sigma}(t)q(t)y^{\sigma}=0 \tag{21}$$

is nonoscillatory on $[T_2,\infty)$. Consequently, since $\frac{1}{2}\mu a^{\Delta}=\frac{1}{2}(a^{\sigma}-a)$, we have

$$p_1=p\left(a+\frac{1}{2}\mu a^{\Delta}\right)=p\left(\frac{a+a^{\sigma}}{2}\right)\leq p,$$

so by the Sturm Comparison Theorem it follows that (2) is also nonoscillatory.

Therefore, we need only show that $G(t)\geq 0$ for all large t. Now by the assumptions (2) and (5) and the first part of the proof, we can choose $T_2\geq T_1$ so that

$$|u(t)|<\min\{\epsilon_0,\epsilon_1\}:=\hat{\epsilon}$$

for $t \in \hat{\mathbb{T}}$ and $t \geq T_2$. Consequently, for $t \in \hat{\mathbb{T}}$ and $t \geq T_2$ we have

$$ap + \mu R = a(p + \mu u) > a(p - \mu\hat{\epsilon}) > 0$$

so that

$$0 < \frac{1}{ap + \mu R} < \frac{1}{a(p - \mu\hat{\epsilon})} \leq \frac{1}{a(p - \mu\epsilon_0)}.$$

It follows that

$$0 \leq \frac{(a^\Delta p)^2}{ap + \mu R} < \frac{(a^\Delta p)^2}{a(p - \mu\epsilon_0)}$$

for $t \in \hat{\mathbb{T}}$, $t \geq T_2$, and therefore, for all $t \in \mathbb{T}$, $t \geq T_2$, we have

$$0 \leq \frac{(a^\Delta p)^2}{ap + \mu R} \leq \frac{(a^\Delta p)^2}{a(p - \mu\epsilon_0)}.$$

By assumption, $G_{\epsilon_0}(t) \geq 0$ for all large t, and hence we have

$$\begin{aligned} 0 \quad &\leq \quad G_{\epsilon_0}(t) = \left(2a^\Delta(t)p(t)\right)^\Delta - \frac{\left(a^\Delta(t)p(t)\right)^2}{a(t)\left(p(t) - \mu(t)\epsilon_0\right)} \\ &\leq \quad \left(2a^\Delta(t)p(t)\right)^\Delta - \frac{\left(a^\Delta(t)p(t)\right)^2}{a(t)p(t) + \mu(t)R(t)} = 4G(t) \end{aligned}$$

for all large t. This completes the proof. □

Proof of Corollary 12:

Proof. Let us suppose that (1) is oscillatory on $[\tau, \infty)$, but that (2) is nonoscillatory on $[\tau, \infty)$. Letting y be a nonoscillatory solution of (2) on $[\tau, \infty)$, we make the Riccati substitution $v(t) = \frac{p(t)y^\Delta(t)}{y(t)}$ and obtain the Riccati equation

$$v^\Delta(t) + a^\sigma(t)q(t) + \frac{v^2(t)}{p(t) + \mu(t)v(t)} = 0, \tag{22}$$

for $t \geq T_1$, for some $T_1 \geq \tau$, and $p(t) + \mu(t)v(t) > 0$, $t \geq T_1$. As in the proof of Theorem 11 it follows that $\lim_{k\to\infty} v(t_k) = 0$ for any sequence $\{t_k\} \subset \hat{\mathbb{T}}$ with $t_k \to \infty$ (using (ii), (iv), and (v)). Let $b(t) = \frac{1}{a(t)}$. Then $b^\Delta(t) = \frac{-a^\Delta(t)}{a(t)a^\sigma(t)}$

and we have

$$\begin{aligned} 2b(t)+\mu(t)b^{\Delta}(t) &= \frac{2}{a(t)}-\frac{\mu(t)a^{\Delta}(t)}{a(t)a^{\sigma}(t)} \\ &= \frac{1}{a(t)a^{\sigma}(t)}(2a^{\sigma}(t)-\mu(t)a^{\Delta}(t)) \\ &\leq \frac{1}{a(t)a^{\sigma}(t)}(2a(t)a^{\sigma}(t))=2 \end{aligned}$$

(since by condition (i) of Corollary 12, $2aa^{\sigma} \geq 2a^{\sigma}-\mu a^{\Delta}$). Therefore, condition (i) of Theorem 11 holds with a replaced by $b=\frac{1}{a}$.
Finally, we can show that condition (iii) of Theorem 11 holds, (again with a replaced by $b=\frac{1}{a}$.) To see this, notice that with $b=\frac{1}{a}$, then, as in the proof of Theorem 11, with the substitution $S=bv$ in equation (22) we obtain the equation

$$S^{\Delta}+q(t)+\frac{(S-\frac{1}{2}b^{\Delta}(t)p(t))^2-\frac{1}{4}(b^{\Delta}(t)p(t))^2}{b(t)p(t)+\mu(t)S}=0.$$

Next with the substitution $\hat{S}(t):=S(t)-\frac{1}{2}b^{\Delta}(t)p(t)$ we have

$$\hat{S}^{\Delta}+q(t)+\frac{\hat{S}^2}{\hat{p_1}(t)+\mu(t)\hat{S}(t)}+H(t)=0,$$

where $\hat{p_1}(t)=b(t)p(t)+\frac{1}{2}\mu(t)b^{\Delta}(t)p(t)$, and

$$H(t):=\frac{1}{2}\left(b^{\Delta}(t)p(t)\right)^{\Delta}-\frac{\left(b^{\Delta}(t)p(t)\right)^2}{4\left(b(t)p(t)+\mu(t)S(t)\right)}.$$

We need to show that $H(t)\geq 0$, as in Theorem 11. If in the definition of $G_{\epsilon_0}(t)$ we replace a by $b=\frac{1}{a}$, then we find that $G_{\epsilon_0}(t)=-H_{\epsilon_0}(t)$. We may now easily show that $4H(t)\geq -H_{\epsilon_0}(t)\geq 0$ holds, for all large t, as in the proof of Theorem 11, and, as in Theorem 11, this implies that (1) is nonoscillatory (since it is the same as $(p(t)y^{\Delta})^{\Delta}+b^{\sigma}(t)(a^{\sigma}(t)q(t))y^{\sigma}=0$). This contradiction establishes the result. □

4. EXAMPLES AND REMARKS

We begin this section with several examples showing the independence of the above criteria.

Example 14. Let $r > 1$. Consider the time scale

$$\mathbb{T} = r^{\mathbb{N}_0} := \left\{ r^k \,:\, k \in \mathbb{N}_0 \right\}.$$

In this case, $\sigma(t) = rt$, $\mu(t) = (r-1)t$ for all $t \in \mathbb{T}$, and any dynamic equation on the time scale $r^{\mathbb{N}_0}$ is called an *r-difference equation*. Let

$$a(t) = \frac{1}{t^2}, \quad p(t) = t \quad \text{and} \quad q(t) = \frac{\gamma \ln r}{(r-1)t \ln t \ln(rt)} + \frac{\lambda(-1)^{N(t)}}{t \ln t},$$

where γ, λ are real constants and $N(t) := \ln t / \ln r \in \mathbb{N}_0$. Observe that $q(t)$ is not eventually of one sign for $\lambda \neq 0$. Since $(\ln t)^\Delta = \frac{\ln r}{(r-1)t}$, it follows that we have

$$\int_1^t \frac{1}{p(s)} \Delta s = \int_1^t \frac{1}{s} \Delta s = \frac{(r-1)\ln t}{\ln r} \to \infty$$

as $t \to \infty$ and so (9) holds. Further,

$$\left(p(t) \left(\frac{1}{a(t)} \right)^\Delta \right)^\Delta = (r+1)^2 t > 0$$

for $t \in \mathbb{T}$, so condition (iii) of Corollary 10 fails to hold. Note that this condition is not even satisfied for any $a(t) = t^{-\omega}$, $\omega > 0$. On the other hand, we have

$$2a(t)\left(a^\Delta(t)\right)^\Delta (p(t) - \mu(t)\epsilon_0) - \left(a^\Delta(t)p(t)\right)^2 = \frac{(r+1)^2}{r^4 t^4}[1 - 2(r-1)\epsilon_0] \geq 0$$

for $0 < \epsilon_0 \leq 1/[2(r-1)]$, and condition (iii) of Theorem 11 is satisfied, and

$$a^\Delta(t) = -\frac{r+1}{r^2 t^3} < 0,$$

so (iii) of Theorem 7 holds. If $0 < \epsilon_1 < 1/(r-1) \leq M$, then $\epsilon_1 \mu(t) < p(t) \leq M\mu(t)$ for all $t \in \mathbb{T}$, so conditions (ii) and (v) of Theorem 11 hold. Breaking up the integral and using the identity $\int_t^{\sigma(t)} f(s)\Delta s = \mu(t) f(t)$ we get

$$\int_t^\infty q(s)\Delta s = \frac{\gamma}{\ln t} + (r-1)\lambda(-1)^{N(t)} \left[\frac{1}{\ln t} - \frac{1}{\ln(rt)} + \frac{1}{\ln(r^2 t)} - \dots \right].$$

Hence

$$\gamma - (r-1)\lambda < \ln t \int_t^\infty q(s)\Delta s < \gamma + (r-1)\lambda.$$

In [9] it was proved that equation (1) is nonoscillatory provided

$$\lim_{t\to\infty} \frac{\mu(t)\frac{1}{p(t)}}{\int_\tau^t \frac{1}{p(s)}\Delta s} = 0 \tag{23}$$

and

$$-\frac{3}{4} < \liminf_{t\to\infty} \mathcal{A}(t) \le \limsup_{t\to\infty} \mathcal{A}(t) < \frac{1}{4},$$

where

$$\mathcal{A}(t) := \left(\int_1^t \frac{1}{p(s)}\Delta s\right)\left(\int_t^\infty q(s)\Delta s\right).$$

We have

$$\frac{\mu(t)\frac{1}{p(t)}}{\int_1^t \frac{1}{p(s)}\Delta s} = \frac{\ln r}{\ln t}$$

and so condition (23) is satisfied. Further,

$$\frac{r-1}{\ln r}[\gamma - (r-1)\lambda] < \mathcal{A}(t) < \frac{r-1}{\ln r}[\gamma + (r-1)\lambda].$$

Set

$$\alpha = \frac{\gamma(r-1)}{\ln r} \quad \text{and} \quad \beta = \frac{\lambda(r-1)^2}{\ln r}.$$

If $\alpha \ge \beta > 0$ and $\alpha + \beta < 1/4$, then (1) is nonoscillatory and Theorem 7 or Theorem 11 can be applied to show that (3) and (2), respectively, are nonoscillatory. If $0 < \beta < -\alpha$ and $\alpha - \beta > -3/4$, then (8) fails to hold, equation (1) is nonoscillatory and in this case, only Theorem 11 can be applied.

Example 15. (i) Let $\mathbb{T} = \mathbb{Z}$, $a(t) = 1/\sqrt{t}$ and $p(t) = \sqrt{t} + \sqrt{t+1}$. Then condition (v) of Theorem 11 fails to hold since $p(t)$ is unbounded. Theorem 7 (for $q(t)$ satisfying (8)) or Corollary 10 can be applied since

$$\sum_{t=\tau}^\infty \frac{1}{p(t)} = \sum_{t=\tau}^\infty \frac{1}{\sqrt{t}+\sqrt{t+1}} = \infty,$$

$$\Delta a(t) = \frac{\sqrt{t}-\sqrt{t+1}}{\sqrt{t(t+1)}} < 0$$

and

$$\Delta\left(p(t)\Delta\left(\frac{1}{a(t)}\right)\right)=\Delta\left[\left(\sqrt{t+1}+\sqrt{t}\right)\left(\sqrt{t+1}-\sqrt{t}\right)\right]=0.$$

(ii) Let $\mathbb{T}=\mathbb{Z}$, $a(t)=t^{-2}$ and $p(t)=(2t+1)^{-1}$. Then condition (ii) of Theorem 11 fails to hold since $p(t)\to 0$ as $t\to\infty$. Theorem 7 (for $q(t)$ satisfying (8)) or Corollary 10 can be applied since

$$\sum_{t=\tau}^{\infty}\frac{1}{p(t)}=\sum_{t=\tau}^{\infty}(2t+1)=\infty,$$

$$\Delta a(t)=\frac{-1-2t}{t^2(t+1)^2}<0$$

and

$$\Delta\left(p(t)\Delta\left(\frac{1}{a(t)}\right)\right)=\Delta\left((2t+1)^{-1}(2t+1)\right)=0.$$

(iii) Let $\mathbb{T}=\mathbb{Z}$, $a(t)=\gamma^{-t}$, $\gamma>1$ and $p(t)=\lambda^t$, $\lambda\in(0,1)$. Then condition (ii) of Theorem 11 and condition (ii) of Theorem 7 fail to hold since $p(t)\to 0$ as $t\to\infty$ and

$$\sum_{t=\tau}^{\infty}\frac{1}{p(t)}=\sum_{t=\tau}^{\infty}\lambda^{-t}=\infty,$$

respectively. On the other hand, the assumptions of Corollary 10 are satisfied provided $\gamma\lambda\in(0,1]$ since we have

$$\Delta a(t)=(1-\gamma)\gamma^{-t-1}<0$$

and

$$\Delta\left(p(t)\Delta\left(\frac{1}{a(t)}\right)\right)=(\gamma-1)(\gamma\lambda-1)(\gamma\lambda)^t<0.$$

Notice that only Corollary 10 can be applied in this case.

Following the idea of the above examples, it is not difficult to find examples showing the independence of Theorem 9 and Corollary 12.

Example 16. Let $\mathbb{T} = \mathbb{Z}$, $p(t) = 1$,

$$a(t) = \frac{1}{2t + (-1)^t} \quad \text{and} \quad q(t) = \frac{\gamma}{t(t+1)} + \frac{\lambda(-1)^t}{t}.$$

It is easy to see that $q(t)$ changes sign for $\lambda \neq 0$ and

$$\Delta a(t) = \frac{-2 + 2(-1)^t}{(2t + 2 - (-1)^t)(2t + (-1)^t)} \leq 0.$$

It can also be shown that conditions (iii) from Corollary 10 and (iii) from Theorem 11 fail to hold since

$$\Delta\left(p(t)\Delta\left(\frac{1}{a(t)}\right)\right) = 4(-1)^t$$

and $2a(t)(1 - \epsilon_0)\Delta^2 a(t) - (\Delta a(t))^2$ is equal to a fraction with a positive denominator and a numerator such that the coefficient in the numerator of the highest power t^2 changes sign. Further we have

$$\gamma - \lambda < t \sum_{s=t}^{\infty} q(s) < \gamma + \lambda.$$

Hence, if $\gamma \geq \lambda > 0$ and $\gamma + \lambda < 1/4$, then equation (1) is nonoscillatory and (8) holds, so only Theorem 7 can be applied. We may obtain the same conclusion for the corresponding oscillatory counterparts provided $a(t) = 2t + (-1)^t$ and $\gamma - \lambda > 1/4$ with $\lambda > 0$.

Remark 17. (Case $\mathbb{T} = \mathbb{R}$) (i) In this case, with the assumption that the expression $p(t)a'(t)$ is differentiable, condition (iii) of Theorem 11 is equivalent to

$$2a(t)(p(t)a'(t))' - p(t)(a'(t))^2 \geq 0,$$

while condition (iii) of Corollary 10 takes the form

$$a(t)(p(t)a'(t))' - 2p(t)(a'(t))^2 \geq 0.$$

This shows that Corollary 10 is a consequence of Theorem 11 in this case. This remark holds also for the oscillatory counterparts if $\mathbb{T} = \mathbb{R}$, see [8].

(ii) Theorem 9 (and Corollary 10) do not require $a(t)$ to be nondecreasing (resp. nonincreasing) on $\mathbb{T} = \mathbb{R}$. Indeed, with $a(t) = 1 - 1/t$ and $p(t) =$

$(t-1)^2$ we have an example of an increasing $a(t)$, where Corollary 10 can be applied. This, however, has no "discrete" counterpart since conditions (ii) from Corollary 10 and (v) from Theorem 11 fail to hold when $\mathbb{T}=\mathbb{Z}$. Note that in Theorem 7 (and Corollary 8) the function $a(t)$ is required to be nonincreasing (resp. nondecreasing) on any time scale.

Remark 18. (Repeated application)

(i) A repeated application of Theorem 9 (resp. Corollary 10) gives the following more general result: Let equation (1) be oscillatory (resp. nonoscillatory), and let the functions $a_1(t), a_2(t), \dots, a_n(t)$ satisfy the assumptions of Theorem 9 (resp. of Corollary 10) and let $a(t)=\prod_{i=1}^n a_i(t)$. Then equation (2) is oscillatory (resp. nonscillatory). It is easy to see that this result is indeed more general; e.g., let $\mathbb{T}=\mathbb{Z}$, $a_1(t)=a_2(t)=t^{-1}$ and $p(t)=1$. The functions $a_1(t), a_2(t)$ satisfy all the assumptions of Corollary 10, but condition (iii) fails to hold for $a(t)=t^{-2}$. Therefore, an iteration (repeated application) gives a better result. Note that the (weaker) assumption (iii) of Theorem 11 is satisfied directly for $a(t)=t^{-2}$, however, but to apply this theorem directly, one needs an additional restriction on $q(t)$.

(ii) Theorem 11 can also be applied repeatedly for monotonic functions $a(t)$ but we must show that

$$\limsup_{t\to\infty} \chi(t)\int_\tau^t q(s)\Delta s > -\infty \quad \text{implies} \quad \limsup_{t\to\infty} \chi(t)\int_\tau^t a^\sigma(t) q(s)\Delta s > -\infty.$$

By the time scale version of the second mean value theorem of integral calculus, see [9], there exists $T=T(t)\in\mathbb{T}$ such that

$$\limsup_{t\to\infty} \chi(t)\int_\tau^t a^\sigma(t) q(s)\Delta s \geq \limsup_{t\to\infty} \chi(t)\left[a(\tau)\int_\tau^{T(t)} q(s)\Delta s + a(t)\int_{T(t)}^t q(s)\Delta s\right].$$

The expression on the right-hand side is greater than $-\infty$ since $a(t)$ is bounded and both integrals are of the same type as that in the assumptions.

Remark 19. A closer examination of the proofs shows that all of the statements can be improved in the following way (assuming the same conditions):
Theorem 7: (1) is nonoscillatory implies (11) is nonoscillatory.
Corollary 8: (1) is oscillatory implies (11) is oscillatory.

Theorem 9: (1) is oscillatory implies (12) is oscillatory.
Corollary 10: (1) is nonoscillatory implies (12) is nonoscillatory.
Theorem 11: (1) is nonoscillatory implies (21) is nonoscillatory.
Corollary 12: (1) is oscillatory implies (21) is oscillatory.

Our statements in the second section then follow from the above by virtue of the Sturm-Picone Comparison Theorem.

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