Oscillation Criteria for Second Order Matrix Dynamic Equations on a Time Scale

Lynn Erbe and Allan Peterson
Department of Mathematics and Statistics, University of Nebraska-Lincoln
Lincoln, NE 68588-0323
lerbe@math.unl.edu
apeterso@math.unl.edu

Abstract

We obtain oscillation criteria for a second order self-adjoint matrix differential equation on a measure chain in terms of the eigenvalues of the coefficient matrices and the graininess function. We illustrate our results with some nontrivial examples.

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In this paper we are concerned with the *self-adjoint* second order matrix differential equation

$$LX(t) = \left[P(t)X^{\Delta}(t)\right]^{\Delta} + Q(t)X^{\sigma}(t) = 0$$

on a time scale $\mathbb{T}$ and the associated *Riccati equation*

$$RZ(t) = Z^{\Delta}(t) + Q(t) + F(t) = 0.$$

where

$$F(t) := Z(t)\left[P(t) + \mu(t)Z(t)\right]^{-1} Z(t). \tag{1}$$

Here by a *time scale* (*measure chain*) $\mathbb{T}$ we just mean a nonempty closed subset of $\mathbb{R}$.
Definition Let $\mathbb{T}$ be a time scale and define the *forward jump operator* $\sigma(t)$ at t for $t < \sup \mathbb{T}$, $t \in \mathbb{T}$ by

$$\sigma(t) := inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the *backward jump* operator $\rho(t)$ at t for $t > \inf \mathbb{T}$ by

$$\rho(t) := sup\{\tau < t : \tau \in \mathbb{T}\},$$

for all $t \in \mathbb{T}$. We assume thoughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered.* If $\sigma(t) = t$ we say t is *right-dense*, while if $\rho(t) = t$ we say t is *left-dense.* A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous provided f is right-continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. Finally, if $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered we let $\mathbb{T}^\kappa := \mathbb{T} \backslash \{\sup \mathbb{T}\}$. Otherwise we let $\mathbb{T}^\kappa := \mathbb{T}$. We shall also use the notation $\mu(t) := \sigma(t) - t$ which is called the graininess function.

Thoughout this paper we make the blanket assumption that a $\leq$ b are points in $\mathbb{T}$ and $\sigma^2(b) \in \mathbb{T}$.

Definition Define the *interval* in $\mathbb{T}$

$$[a, b] := \{t \in \mathbb{T} \ \ such \ \ that \ \ a \leq t \leq b\}.$$

Other types of intervals are defined similarly.

We are concerned with calculus on measure chains which is a unified approach to continuous and discrete calculus. An excellent introduction is given by S. Hilger [13]. Agarwal and Bohner [1] refer to it as calculus on time scales. Other papers in this area include Agarwal and Bohner [3], Agarwal, Bohner, and Wong [4], and Hilger and Erbe [11], and Erbe and Peterson [12]. The oscillation results given here generalize the oscillation results of Peterson and Ridenhour [16] for a discrete self-adjoint difference equation with step size one. See [2], [5], and [15] for related results.

Definition Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$ such that $t < \sup \mathbb{T}$, then we define x^Δ(t) to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all s $\in$ U. We call $x^\Delta(t)$ the *delta derivative* of x(t).

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$, $t < \sup \mathbb{T}$, and t is right-scattered, then

$$x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T}$=$\mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, then

$$x^\Delta(t) = \Delta x(t) := x(t + 1) - x(t).$$

We assume throughout that the coefficient matrices satisfy $P(t) > 0$ $(positive\ definite)$ for $t \in \mathbb{T}^{\kappa^2} := (\mathbb{T}^\kappa)^\kappa$ and $Q(t) = Q^*(t)$ ($Q(t)$ is Hermitian) for $t \in \mathbb{T}^\kappa$. If a matrix function $X(t)$ is differentiable, then a formula that we will use frequently is $X^\sigma(t) = X(t) + \mu(t)X^\Delta(t)$ for $t \in \mathbb{T}^\kappa$.

Definition Let $\mathbb{D}$ denote the set of all $n \times n$ matrix functions $X(t)$ defined on $\mathbb{T}$ such that $X(t)$ is delta differentiable on $\mathbb{T}^\kappa$ and $(P(t)X^\Delta(t))^\Delta$ is right-dense continuous on $\mathbb{T}^{\kappa^2}$.

Definition If $X, Y \in \mathbb{D}$, then we define the *Wronskian matrix* of $X(t), Y(t)$ by

$$W[X(t), Y(t)] = X^*(t)P(t)Y^\Delta(t) - [P(t)X^\Delta(t)]^*Y(t)$$

for $t \in T^\kappa$.

In the standard way one can easily prove the following two results.

Theorem 1 *(Lagrange Identity) If $X, Y \in \mathbb{D}$, then*

$$Y^*(\sigma(t))LX(t) - [LY(t)]^* X(\sigma(t)) = (W[X(t), Y(t)])^\Delta$$

for $t \in \mathbb{T}^{\kappa^2}$.

Corollary 2 *(Abel's Formula) If $X(t), Y(t)$ are solutions of $LX(t) = 0$ then*

$$W[X(t), Y(t)] = C$$

for $t \in \mathbb{T}^\kappa$, where C is a constant matrix.

From Abel's formula we get that if $X(t)$ is a solution of $LX(t) = 0$, then

$$W[X(t), X(t)] = C$$

for $t \in \mathbb{T}^\kappa$, where C is a constant matrix. With this in mind we make the following definition.
Definition If $X(t)$ is a solution of $LX(t) = 0$ with

$$W[X(t), X(t)] = 0$$

for $t \in \mathbb{T}^\kappa$, then we say $X(t)$ is a *prepared solution* (*conjoined solution*, *isotropic solution*) of $LX(t) = 0$.

Note that $X(t)$ is a prepared solution of $LX(t) = 0$ iff there is a $t_0 \in \mathbb{T}$ such that

$$W[X(t), X(t)]_{t=t_0} = 0.$$

It is also straight forward to show that a solution $X(t)$ is a prepared solution iff $X^*(t)P(t)X^\Delta(t)$ is Hermitian for all $t \in \mathbb{T}$ iff $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian for some $t_0 \in \mathbb{T}$. We can easily get prepared solutions of $LX(t) = 0$ by taking intitial conditions at $t_0 \in \mathbb{T}$ so that $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian. Also if $X(t)$ is a prepared solution then for any complex number z_0, $Y(t) = z_0 X(t)$ is also a prepared solution. Note however that if $\mathbb{T} = \mathbb{R}$, then $x(t) = e^{it}$ is not a prepared solution of the scalar differential equation $x'' + x = 0$.
In the Sturmian theory for $LX(t) = 0$ the matrix function

$$X^*(\sigma(t))P(t)X(t)$$

is very important. We note the following result (see also Agarwal and Bohner [3]).

Lemma 3 *Let $X(t)$ be a solution of $LX(t) = 0$. If $X(t)$ is a prepared solution of $LX(t) = 0$, then*

$$X^*(\sigma(t))P(t)X(t)$$

is Hermitian for all $t \in \mathbb{T}^\kappa$. Conversely, if there is a $t_0 \in \mathbb{T}^\kappa$ such that $\mu(t_0) = \sigma(t_0) - t_0 > 0$ and $X^*(\sigma(t_0))P(t_0)X(t_0)$ is Hermitian, then $X(t)$ is a prepared solution of $LX(t) = 0$. Also if $X(t)$ is a nonsingular prepared solution, then $P(t)X(\sigma(t))X^{-1}(t)$ $\{P(t)X(t)X^{-1}(\sigma(t))\}$ is Hermitian for all $t \in \mathbb{T}^\kappa$. Finally if $X(t)$ is a nonsingular prepared solution, then $P(t)X(\sigma(t))X^{-1}(t)$ and the Riccati function $Z(t) := P(t)X^\Delta(t)X^{-1}(t)$ are Hermitian for all $t \in \mathbb{T}^\kappa$.

Proof: Let $X(t)$ be a solution of $LX(t)=0$. If $X(t)$ is a prepared solution of $LX(t)=0$, then

$$\begin{aligned}X^*(\sigma(t))P(t)X(t) &= [X(t)+\mu(t)X^\Delta(t)]^*P(t)X(t)\\ &= X^*(t)P(t)X(t)+\mu(t)[X^\Delta(t)]^*P(t)X(t)\\ &= X^*(t)P(t)X(t)+\mu(t)X^*(t)P(t)X^\Delta(t)\end{aligned}$$

for $t \in \mathbb{T}^\kappa$. It follows that $X^*(\sigma(t))P(t)X(t)$ is Hermitian for $t \in \mathbb{T}^\kappa$. Conversely, assume there is a $t_0 \in \mathbb{T}^\kappa$ such that $\mu(t_0)>0$ and

$$X^*(\sigma(t_0))P(t_0)X(t_0)$$

is Hermitian. Then

$$\begin{aligned}X^*(t_0)P(t_0)X^\Delta(t_0) &= X^*(t_0)P(t_0)\frac{X(\sigma(t_0))-X(t_0)}{\mu(t_0)}\\ &= \frac{1}{\mu(t_0)}[X^*(t_0)P(t_0)X(\sigma(t_0))-X^*(t_0)P(t_0)X(t_0)]\\ &= \frac{1}{\mu(t_0)}[X^*(\sigma(t_0))P(t_0)X(t_0)-X^*(t_0)P(t_0)X(t_0)].\end{aligned}$$

It follows that $X^*(t_0)P(t_0)X^\Delta(t_0)$ is Hermitian which implies that $X(t)$ is a prepared solution of $LX(t)=0$. Finally assume that $X(t)$ is a nonsingular prepared solution of $LX(t)=0$. Since $X(t)$ is a prepared solution,

$$X^*(\sigma(t))P(t)X(t)=X(t)^*P(t)X(\sigma(t)) \tag{2}$$

for $t \in \mathbb{T}^\kappa$. Since $X(t)$ is invertible,

$$P(t)X(\sigma(t))X^{-1}(t)=(X^{-1}(t))^*X^*(\sigma(t))P(t)$$

for $t \in \mathbb{T}^\kappa$, which gives us that $P(t)X(\sigma(t))X^{-1}(t)$ is Hermitian for $t \in \mathbb{T}^\kappa$. Also from (2) we get that

$$P(t)X(t)X^{-1}(\sigma(t))=[X^{-1}(\sigma(t))]^*X^*(t)P(t)$$

for $t \in \mathbb{T}^\kappa$. Hence $P(t)X(t)X^{-1}(\sigma(t))$ is also Hermitian for $t \in \mathbb{T}^\kappa$. Finally, assume $X(t)$ is a nonsingular prepared solution of $LX(t)=0$ and let

$$Z(t) := P(t)X^\Delta(t)X^{-1}(t).$$

Since $X(t)$ is prepared,

$$X^*(t)P(t)X^\Delta(t)=[P(t)X^\Delta(t)]^*X(t)$$

for all $t \in \mathbb{T}^\kappa$. Since $X(t)$ is invertible

$$P(t)X^\Delta(t)X^{-1}(t)=[X^{-1}(t)]^*[P(t)X^\Delta(t)]^*$$

for $t \in \mathbb{T}^\kappa$. Hence

$$Z(t)=Z^*(t)$$

for $t \in \mathbb{T}^\kappa$, which is the desired result. □

Lemma 4 *Assume that $X(t)$ is a prepared solution of $LX = 0$ on $\mathbb{T}$. Then the following are equivalent:*
(a) $X^*(\sigma(t))P(t)X(t) > 0$ on $\mathbb{T}^\kappa$.
(b) X(t) is nonsingular and

$$P(t)X(\sigma(t))X^{-1}(t) > 0$$

on $\mathbb{T}^\kappa$.
(c) X(t) is nonsingular and

$$P(t)X(t)X^{-1}(\sigma(t)) > 0$$

on $\mathbb{T}^\kappa$.

Proof: First note that $X^*(\sigma(t))P(t)X(t) > 0$ on $\mathbb{T}^\kappa$ implies that $X(t)$ is nonsingular on $\mathbb{T}^\kappa$. Since $X(t)$ is a prepared solution we have by Lemma 3

$$X^*(t)P(t)X(\sigma(t)) = X^*(\sigma(t))P(t)X(t)$$

for $t \in \mathbb{T}^\kappa$. It follows that

$$P(t)X(\sigma(t))X^{-1}(t) = [X^*]^{-1}(t)X^*(\sigma(t))P(t)$$

for $t \in \mathbb{T}^\kappa$. This implies that

$$P(t)X(\sigma(t))X^{-1}(t) = \left[X^{-1}(t)\right]^* \left[X^*(\sigma(t))P(t)X(t)\right] X^{-1}(t)$$

for $t \in \mathbb{T}^\kappa$. This last equation implies

$$X^*(\sigma(t))P(t)X(t) > 0$$

for $t \in \mathbb{T}^\kappa$ iff

$$P(t)X(\sigma(t))X^{-1}(t) > 0$$

for $t \in \mathbb{T}^\kappa$. Hence (a) and (b) are equivalent.
The equality

$$X^*(\sigma(t))P(t)X(t) = X^*(\sigma(t))\left[P(t)X(t)X^{-1}(\sigma(t))\right] X(\sigma(t))$$

for $t \in \mathbb{T}^\kappa$, shows that (a) and (c) are equivalent. □

Later in this paper we will be concerned with the oscillation of the equation $LX = 0$. We now define what is meant by oscillation.
Definition Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. We say that $LX = 0$ is *nonoscillatory* on $[a, \infty)$ provided there is a prepared solution $X(t)$ of $LX = 0$ and a $t_0 \in [a, \infty)$ such that

$$X^*(\sigma(t))P(t)X(t) > 0$$

on $[t_0, \infty)$. Otherwise we say $LX = 0$ is *oscillatory* on $[a, \infty)$.

We remark that Theorem 5 and Theorem 6 below have been established independently by Agarwal and Bohner [3]. We have included the statements and proofs of these two theorems for the sake of completeness.

Theorem 5 *The self-adjoint matrix equation* $LX = 0$ *has a prepared solution* $X(t)$ *on* $\mathbb{T}$ *with* $X^*(\sigma(t))P(t)X(t) > 0$ *on* $\mathbb{T}^\kappa$ *iff the Riccati equation* $RZ(t) = 0$ *has a Hermitian solution* $Z(t)$ *on* T^κ *satisfying*

$$P(t) + \mu(t)Z(t) > 0 \tag{3}$$

on $\mathbb{T}^\kappa$*.*

Proof: Assume $X(t)$ is a prepared solution of $LX = 0$ on $\mathbb{T}$ with $X^*(\sigma(t))P(t)X(t) > 0$ on T^κ. By Lemma 4 it follows that $X(t)$ is invertible on T^κ. Also

$$\begin{aligned} X^*(\sigma(t))P(t)X(t) &= [X(t) + \mu(t)X^\Delta(t)]^*P(t)X(t) \\ &= X^*(t)P(t)X(t) + \mu(t)[X^\Delta(t)]^*P(t)X(t) \\ &= X^*(t)P(t)X(t) + \mu(t)X^*(t)P(t)X^\Delta(t) \\ &= X^*(t)\left\{P(t) + \mu(t)[P(t)X^\Delta(t)X^{-1}(t)]\right\}X(t) \\ &= X^*(t)\{P(t) + \mu(t)Z(t)\}X(t) \end{aligned}$$

for $t \in \mathbb{T}^\kappa$. Since $X(t)$ is nonsingular and $X^*(\sigma(t))P(t)X(t) > 0$ on T^κ we get that

$$P(t) + \mu(t)Z(t) > 0$$

for $t \in \mathbb{T}^\kappa$ and hence (3) holds. We make the *Riccati substitution*

$$Z(t) := P(t)X^\Delta(t)X^{-1}(t)$$

for $t \in \mathbb{T}^\kappa$. By Lemma 3, $Z(t)$ is Hermitian for all $t \in \mathbb{T}^\kappa$. By the product rule

$$\begin{aligned} Z^\Delta(t) &= \left[P(t)X^\Delta(t)\right]^\Delta X^{-1}(\sigma(t)) + P(t)X^\Delta(t)\left[X^{-1}(t)\right]^\Delta \\ &= -Q(t)X(\sigma(t))X^{-1}(\sigma(t)) - P(t)X^\Delta(t)X^{-1}(\sigma(t))X^\Delta(t)X^{-1}(t). \end{aligned}$$

Hence

$$Z^\Delta(t) = -Q(t) - Z(t)X(t)X^{-1}(\sigma(t))X^\Delta(t)X^{-1}(t). \tag{4}$$

Note that

$$\begin{aligned} P(t)X^\sigma(t)X^{-1}(t) &= P(t)[X(t) + \mu(t)X^\Delta(t)]X^{-1}(t) \\ &= P(t) + \mu(t)Z(t). \end{aligned}$$

Hence we get the important equation

$$P(t)X^\sigma(t)X^{-1}(t) = P(t) + \mu(t)Z(t) \tag{5}$$

for $t \in \mathbb{T}^\kappa$. It follows from (5) that

$$X(t)X^{-1}(\sigma(t)) = [P(t) + \mu(t)Z(t)]^{-1}P(t).$$

Substituting this into equation (4) we have

$$\begin{aligned} Z^{\Delta}(t) &= -Q(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} P(t)X^{\Delta}(t)X^{-1}(t) \\ &= -Q(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t). \end{aligned}$$

Hence $Z(t)$ is a Hermitian solution of the Riccati equation $RZ = 0$ on $\mathbb{T}^{\kappa}$ and (3) is satisfied for $t \in \mathbb{T}^{\kappa}$. Conversely assume $Z(t)$ is a solution of $RZ = 0$ on $\mathbb{T}^{\kappa}$ and (3) is satisfied for $t \in \mathbb{T}^{\kappa}$. Let $t_0 \in \mathbb{T}^{\kappa}$ and let $X(t)$ be the solution of the initial value problem (IVP)

$$\begin{aligned} X^{\Delta}(t) &= P^{-1}(t)Z(t)X(t) \\ X(t_0) &= I. \end{aligned}$$

This IVP has a unique solution $X(t)$ which exists on all of $\mathbb{T}$ because

$$I + \mu(t)P^{-1}(t)Z(t) = P^{-1}(t)\left[P(t)+\mu(t)Z(t)\right]$$

is invertible for $t \in \mathbb{T}^{\kappa}$. Note that

$$P(t)X^{\Delta}(t) = Z(t)X(t)$$

for $t \in \mathbb{T}^{\kappa^2}$. Taking the delta derivative of both sides we obtain

$$\begin{aligned} \left[P(t)X^{\Delta}(t)\right]^{\Delta} &= Z^{\Delta}(t)X^{\sigma}(t) + Z(t)X^{\Delta}(t) \\ &= \left\{-Q(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)\right\} X^{\sigma}(t) + Z(t)X^{\Delta}(t) \\ &= -Q(t)X^{\sigma}(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &+ Z(t)X^{\Delta}(t) \\ &= -Q(t)X^{\sigma}(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &+ Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1}\left[P(t)+\mu(t)Z(t)\right] X^{\Delta}(t) \\ &= -Q(t)X^{\sigma}(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &+ Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1}\left[P(t)X^{\Delta}(t)+\mu(t)Z(t)X^{\Delta}(t)\right] \\ &= -Q(t)X^{\sigma}(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &+ Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1}\left[Z(t)X(t)+\mu(t)Z(t)X^{\Delta}(t)\right] \\ &= -Q(t)X^{\sigma}(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &+ Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)\left[X(t)+\mu(t)X^{\Delta}(t)\right] \\ &= -Q(t)X^{\sigma}(t) - Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &+ Z(t)\left[P(t)+\mu(t)Z(t)\right]^{-1} Z(t)X^{\sigma}(t) \\ &= -Q(t)X^{\sigma}(t) \end{aligned}$$

for $t \in \mathbb{T}^{\kappa^2}$. Hence $X(t)$ is a solution of $LX(t) = 0$ on $\mathbb{T}$. Also

$$X^{*}(\sigma(t))P(t)X(t) = X^{*}(t)[P(t)+\mu(t)Z(t)]X(t)$$

on $\mathbb{T}^\kappa$ and $X(t)$ is nonsingular we have

$$X^*(\sigma(t))P(t)X(t) > 0.$$

$\square$

Theorem 6 *Assume $Z(t)$ is a solution of the Riccati equation $RZ(t) = 0$ on $\mathbb{T}$ with $P(t) + \mu(t)Z(t) > 0$ on $\mathbb{T}^\kappa$ and assume $U(t)$ is delta differentiable on $\mathbb{T}^\kappa$. Then*

$$[U^*(t)Z(t)U(t)]^\Delta = \{U^{*\Delta}(t)P(t)U^\Delta(t) - U^*(\sigma(t))Q(t)U(\sigma(t))\} - H^*(t)H(t)$$

for $t \in \mathbb{T}^\kappa$, where

$$H(t) := [P(t) + \mu(t)Z(t)]^{\frac{1}{2}} U^\Delta(t) - [P(t) + \mu(t)Z(t)]^{-\frac{1}{2}} Z(t)U(\sigma(t))$$

for $t \in \mathbb{T}^\kappa$.

Proof: By the product rule

$$\begin{aligned}
[U^*(t)Z(t)U(t)]^\Delta &= U^*(\sigma(t))Z^\Delta(t)U(\sigma(t)) + U^*(\sigma(t))Z(t)U^\Delta(t) + U^{*\Delta}(t)Z(t)U(t) \\
&= U^*(\sigma(t))\left\{-Q(t) - Z(t)\left[P(t) + \mu(t)Z(t)\right]^{-1} Z(t)\right\} U(\sigma(t)) \\
&+ U^*(\sigma(t))Z(t)U^\Delta(t) + U^{*\Delta}(t)Z(t)\left[U(\sigma(t)) - \mu(t)U^\Delta(t)\right] \\
&= \left\{U^{*\Delta}(t)P(t)U^\Delta(t) - U^*(\sigma(t))Q(t)U(\sigma(t))\right\} \\
&- U^{*\Delta}(t)\left[P(t) + \mu(t)Z(t)\right]U^\Delta(t) + U^*(\sigma(t))Z(t)U^\Delta(t) + U^{*\Delta}(t)Z(t)U(\sigma(t)) \\
&- U^*(\sigma(t))Z(t)\left[P(t) + \mu(t)Z(t)\right]^{-1} Z(t)U(\sigma(t)) \\
&= \left\{U^{*\Delta}(t)P(t)U^\Delta(t) - U^*(\sigma(t))Q(t)U(\sigma(t))\right\} \\
&- \left\{U^{*\Delta}(t)\left[P(t) + \mu(t)Z(t)\right]^{\frac{1}{2}} - U^*(\sigma(t))Z(t)\left[P(t) + \mu(t)Z(t)\right]^{-\frac{1}{2}}\right\} \\
&\cdot \left\{\left[P(t) + \mu(t)Z(t)\right]^{\frac{1}{2}} U^\Delta(t) - \left[P(t) + \mu(t)Z(t)\right]^{-\frac{1}{2}} Z(t)U(\sigma(t))\right\} \\
&= \{U^{*\Delta}(t)P(t)U^\Delta(t) - U^*(\sigma(t))Q(t)U(\sigma(t))\} - H^*(t)H(t).
\end{aligned}$$

$\square$

We now introduce some notation that we will use in the remainder of this paper. If A is an $n \times n$ Hermitian matrix, then we will let $\lambda_i(A)$ denote the i^{th} eigenvalue of A so that

$$\lambda_{\max}(A) = \lambda_1(A) \geq \cdots \geq \lambda_n(A) = \lambda_{\min}(A).$$

The *trace* of a matrix A is denoted by

$$tr(A) := \sum_{i=1}^{n} a_{ii}.$$

We shall make frequent use of Weyl's theorem which says if A and B are Hermitian matrices, then

$$\lambda_i(A) + \lambda_{\max}(B) \geq \lambda_i(A + B) \geq \lambda_i(A) + \lambda_{\min}(B).$$

We now state and prove an oscillation theorem for the matrix equation

$$X^{\Delta\Delta}(t)+Q(t)X(\sigma(t))=0. \tag{6}$$

In the results to follow, we shall assume that the graininess function $\mu(t)\not\equiv 0$ for large t. Otherwise, if $\mu(t)\equiv 0$, then the self-adjoint dynamic equation reduces to an ordinary differential equation, for which many criteria are known (see Butler, Erbe, and Mingarelli [8] and Byers, Harris, and Kwong [9]).

Theorem 7 *Assume* $a\in\mathbb{T}$ *and* $\sup\mathbb{T}=\infty$. *Assume for any* $t_0\in[a,\infty)$ *there exists* $t_0\le a_0<b_0$ *such that* $\mu(a_0)>0$, $\mu(b_0)>0$ *and*

$$\lambda_{\max}\left(\int_{a_0}^{b_0}Q(t)\Delta t\right)\ge\frac{1}{\mu(a_0)}+\frac{1}{\mu(b_0)}. \tag{7}$$

Then equation (6) *is oscillatory on* $[a,\infty)$.

Proof: Assume equation (6) is nonoscillatory on $[a,\infty)$. Then there is a $t_0\in[a,\infty)$ and a prepared solution $X(t)$ of (6) satisfying

$$X^*(\sigma(t))X^{\Delta}(t)>0$$

on $[t_0,\infty)$. We make the Riccati substitution

$$Z(t)=X^{\Delta}(t)X^{-1}(t)$$

for $t\in[t_0,\infty)$, then by Theorem 5 we get that

$$I+\mu(t)Z(t)>0$$

on $[t_0,\infty)$ and $Z(t)$ satisfies the Riccati equation $RZ=0$ on $[t_0,\infty)$. By hypothesis there exist $t_0\le a_0<b_0$ such that $\mu(a_0)>0$, $\mu(b_0)>0$ and inequality (7) holds. Integrating both sides of the Riccati equation from a_0 to $t > a_0$ we obtain

$$\begin{aligned}
Z(t) &= Z(a_0) - \int_{a_0}^{t} Q(s)\Delta s - \int_{a_0}^{t} Z(s)[I+\mu(s)Z(s)]^{-1}Z(s)\Delta s \\
&= Z(a_0) - \int_{a_0}^{t} Q(s)\Delta s - \int_{a_0}^{\sigma(a_0)} Z(s)[I+\mu(s)Z(s)]^{-1}Z(s)\Delta s \\
&- \int_{\sigma(a_0)}^{t} Z(s)[I+\mu(s)Z(s)]^{-1}Z(s)\Delta s \\
&= Z(a_0) - \mu(a_0)Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}Z(a_0) - \int_{a_0}^{t} Q(s)\Delta s \\
&- \int_{\sigma(a_0)}^{t} Z(s)[I+\mu(s)Z(s)]^{-1}Z(s)\Delta s \\
&= Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}[I+\mu(a_0)Z(a_0)-\mu(a_0)Z(a_0)] - \int_{a_0}^{t} Q(s)\Delta s \\
&- \int_{\sigma(a_0)}^{t} Z(s)[I+\mu(s)Z(s)]^{-1}Z(s)\Delta s \\
&= Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1} - \int_{a_0}^{t} Q(s)\Delta s \\
&\quad - \int_{\sigma(a_0)}^{t} Z(s)[I+\mu(s)Z(s)]^{-1}Z(s)\Delta s.
\end{aligned}$$

It follows that

$$Z(t) + \int_{a_0}^{t} Q(s)\Delta s \leq Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}. \tag{8}$$

Now let U be a unitary matrix (so $U^*U = I$) such that

$$Z(a_0) = U^*DU,$$

where

$$D := \mathrm{diag}(d_1, \cdots, d_n),$$

where $d_i = \lambda_i(Z(a_0))$ is the i^{th} eigenvalue of $Z(a_0)$, $i = 1, 2, \cdots$. Consider

$$\begin{aligned}
Z(t_0)[I+\mu(a_0)Z(a_0)]^{-1} &= U^*DU[I+\mu(a_0)U^*DU]^{-1} \\
&= U^*DU\left\{U^*[I+\mu(a_0)D]U\right\}^{-1} \\
&= U^*D[I+\mu(a_0)D]^{-1}U.
\end{aligned}$$

Since $I + \mu(a_0)Z(a_0) > 0$ implies that

$$1 + \mu(a_0)d_i > 0$$

and $h(x) := \frac{x}{1+\mu(a_0)x}$ is increasing when $1 + \mu(a_0)x > 0$, it follows that

$$\begin{aligned}\lambda_i\left(Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}\right) &= \lambda_i\left(D[I+\mu(a_0)D]\right)\\ &= \frac{d_i}{1+\mu(a_0)d_i}.\end{aligned}$$

Using $h(x) = \frac{x}{1+\mu(a_0)x} < \frac{1}{\mu(a_0)}$ when $1 + \mu(a_0)x > 0$ we get

$$\lambda_i\left(Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}\right) = \frac{d_i}{1+\mu(a_0)d_i} < \frac{1}{\mu(a_0)}. \tag{9}$$

Hence from equation (8) we obtain

$$\lambda_i\left(Z(t) + \int_{a_0}^{t} Q(s)\Delta s\right) \leq \lambda_i\left(Z(a_0)[I+\mu(a_0)Z(a_0)]^{-1}\right) < \frac{1}{\mu(a_0)}.$$

Applying Weyl's inequality we get

$$\frac{1}{\mu(a_0)} > \lambda_{\max}\left(Z(t) + \int_{a_0}^{t} Q(s)\Delta s\right) \geq \lambda_{\max}\left(\int_{a_0}^{t} Q(s)\Delta s\right) + \lambda_{\min}\left(Z(t)\right)$$

Letting $t = b_0$ and rearranging terms in this inequality we get

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(s)\Delta s\right) < \frac{1}{\mu(a_0)} - \lambda_{\min} Z(b_0).$$

Since $I + \mu(b_0)Z(b_0) > 0$ implies $\lambda_{\min}\left(Z(b_0)\right) > -\frac{1}{\mu(b_0)}$ we get that

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(s)\Delta s\right) < \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}$$

which is a contradiction. $\square$

Example 8 *Assume the scalar functions* $q_i : \mathbb{T}^\kappa \to \mathbb{R}$, $1 \leq i \leq n$ *are rd-continuous on* $\mathbb{T} := \cup_{k=0}^{\infty}[k, k+\frac{1}{2}]$ *and assume for each* $t_0 \in [0,\infty)$ *there is a* $k_0 \in \mathbb{N}$, *and a* $l_0 \in \mathbb{N}$, *such that* $k_0 \geq t_0$ *and*

$$\sum_{j=1}^{l_0} \int_{k_0+j}^{k_0+j+\frac{1}{2}} q_1(t)dt + \frac{1}{2}\sum_{j=0}^{l_0-1} q_1(k_i + j + \frac{1}{2}) \geq 4. \tag{10}$$

Further assume that for $2 \leq j \leq n$,

$$\int_{k_0+\frac{1}{2}}^{k_0+l_0+\frac{1}{2}} q_1(t)dt \geq \int_{k_0+\frac{1}{2}}^{k_0+l_0+\frac{1}{2}} q_j(t)dt.$$

Then, if

$$Q(t) := diag\left(q_1(t), q_2(t), \cdots, q_n(t)\right),$$

it follows that the matrix dynamic equation

$$X^{\Delta\Delta} + Q(t)X^{\sigma} = 0$$

is oscillatory on $\mathbb{T}$. *To see this follows from Theorem 7 let* $t_0 \in \mathbb{T}$ *be given and let*

$$a_0 := k_0 + \frac{1}{2} \quad \textit{and} \quad b_0 := k_0 + l_0 + \frac{1}{2}.$$

Note that a_0 *and* b_0 *are right scattered and*

$$\begin{aligned} \lambda_{\max} \int_{a_0}^{b_0} Q(t)\Delta t &= \int_{a_0}^{b_0} q_1(t)\Delta t \\ &= \sum_{j=1}^{l_0} \int_{k_0+j}^{k_0+j+\frac{1}{2}} q_1(t)\Delta t + \sum_{j=0}^{l_0-1} \int_{k_0+j+\frac{1}{2}}^{\sigma(k_0+j+\frac{1}{2})} q_1(t)\Delta t \\ &= \sum_{j=1}^{l_0} \int_{k_0+j}^{k_0+j+\frac{1}{2}} q_1(t)dt + \frac{1}{2}\sum_{j=0}^{l_0-1} q_1(k_0+j+\frac{1}{2}). \end{aligned}$$

Since

$$\frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)} = 4$$

this result follows from Theorem 7. Note there is no requirement on

$$\liminf_{t\to\infty} \int_{t_0}^{t} q_j(\tau)\Delta\tau$$

for $1 \le j \le n$.

Any dynamic equation on the measure chain

$$\mathbb{T}_q := \{q^k : k = 0, 1, 2, \cdots\}$$

where $q > 1$ is called a q-difference equation. See J. P. Bézivin [6], W. J. Trijtzinsky [17], and C. Zhang [18] for important papers concerning q-difference equations. Also see M. Bohner and D. A. Lutz [7] for some additional results for q-difference equations. In the next example we show that a certain q-difference equation is oscillatory.

Example 9 *In this example we show that the* q*-difference equation*

$$x^{\Delta\Delta}(t) + \frac{c}{q(q-1)t^2}x^{\sigma}(t) = 0, \tag{11}$$

where $c > 1$ *is a constant, is oscillatory on* $\mathbb{T} := q^{N_0}$ *where* $q > 1$ *is a constant. Let* $t_0 \in [0, \infty)$ *be given and pick a positive integer* k_0 *so that* $a_0 := q^{k_0} > t_0$. *Let* $\tilde{q}(t) := \frac{c}{q(q-1)t^2}$ *for* $t \in \mathbb{T}$ *and* *consider*

$$\begin{aligned}\int_{a_0}^{\infty} \tilde{q}(t)\Delta t &= \int_{q^{k_0}}^{\infty} \tilde{q}(t)\Delta t \\ &= \sum_{j=k_0}^{\infty} \tilde{q}(q^j)\mu(q^j) = \sum_{j=k_0}^{\infty} \tilde{q}(q^j)(q-1)q^j \\ &= \sum_{j=k_0}^{\infty} \frac{c}{q(q-1)q^{2j}}(q-1)q^j = \frac{c}{q}\sum_{j=k_0}^{\infty}\frac{1}{q^j} \\ &= \frac{c}{q^{k_0+1}}\sum_{j=0}^{\infty}\left(\frac{1}{q}\right)^j = \frac{c}{(q-1)q^{k_0}}.\end{aligned}$$

Using this, $c > 1$*, and the fact that* $\lim_{k\to\infty}\mu(t_k) = \infty$ *we can pick* $b_0 \in (a_0, \infty)$ *sufficiently large so that*

$$\begin{aligned}\int_{a_0}^{b_0} \tilde{q}(t)\Delta t &> \frac{1}{(q-1)q^{k_0}} + \frac{1}{\mu(b_0)} \\ &= \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}.\end{aligned}$$

Hence from Theorem 7 we get that the q-difference equation (11) *is oscillatory on* $\mathbb{T}$.

Corollary 10 *Assume* $a \in \mathbb{T}$ *and* $\sup \mathbb{T} = \infty$*. If there exists a sequence* $\{t_k\}_{k=1}^{\infty} \subset [a, \infty)$ *such that* $\lim_{k\to\infty} t_k = \infty$ *with* $\mu(t_k) \geq K > 0$ *for some* $K > 0$ *and such that*

$$\limsup_{t_k\to\infty} \lambda_{\max}\left(\int_a^{t_k} Q(s)\Delta s\right) = +\infty, \tag{12}$$

then equation (6) *is oscillatory. To see this let* $t_0 \in [a, \infty)$*. Choose* k_0 *sufficiently large so that* $a_0 := t_{k_0} \in [t_0, \infty)$*. Using* (12) *we can pick* $k_1 > k_0$ *sufficiently large so that with* $b_0 := t_{k_1}$ *we have*

$$\lambda_{\max}\left(\int_{a_0}^{b_0} Q(s)\Delta s\right) \geq \frac{2}{K} \geq \frac{1}{\mu(a_0)} + \frac{1}{\mu(b_0)}.$$

It should be noted that Corollary 10 is an extension to the measure chain situation of the so-called "largest eigenvalue Theorem" which says that the matrix differential system

$$Y'' + Q(t)Y = 0$$

is oscillatory if

$$\lim_{t\to\infty} \lambda_{\max}\left(\int_{t_0}^{t} Q(s)ds\right) = \infty.$$

In the discrete case where $\mathbb{T} = \mathbb{Z}$, one can replace "lim" by "lim sup," as was noted by Chen and Erbe [10] and Peterson and Ridenhour [16]. The result above again illustrates the importance of the graininess function on a sequence $t_k \to \infty$.

We suppose in what follows that there is a sequence $\{t_k\} \subset [a, \infty)$ with $\lim_{k\to\infty} t_k = \infty$ such that $\mu(t_k) > 0$ (in other words we are excluding the case of vanishing graininess, $\mu(t) = 0$ for all large t.

Theorem 11 *Assume* $a \in \mathbb{T}$ *and* $\sup \mathbb{T} = \infty$ *and suppose there is a strictly increasing sequence* $\{t_k\} \subset [a, \infty)$ *such that* $\mu(t_k) > 0$ *for* $k = 1, 2, \cdots$, *with* $\lim_{k\to\infty} t_k = \infty$. *Further assume that there is a sequence* $\{\tau_k\} \subset [a, \infty)$ *such that* $\sigma(\tau_k) > \tau_k \geq \sigma(t_k)$, *for* $k = 1, 2, \cdots$ *such that*

$$\lambda_{\min}\left(\frac{P(t_k)}{\mu(t_k)} + \frac{P(\tau_k)}{\mu(\tau_k)} - \int_{t_k}^{\tau_k} Q(s)\Delta s\right) \leq 0$$

for $k = 1, 2, \cdots$. *Then* $LX(t) = 0$ *is oscillatory on* $[a, \infty)$.

Proof: Assume $LX = 0$ is nonoscillatory on $[a, \infty)$. Then there is a prepared solution $X(t)$ of $LX = 0$ and a $t_0 \in [a, \infty)$ such that

$$X^*(\sigma(t))P(t)X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) := P(t)X^\Delta(t)X^{-1}(t)$$

for $t \in [a, \infty)$. Then by Theorem 5 we get that

$$P(t) + \mu(t)Z(t) > 0$$

on $[t_0, \infty)$ and $Z(t)$ is a Hermitian solution of the Riccati equation $RZ = 0$ on $[t_0, \infty)$. Let $\{t_k\}, \{\tau_k\}$ be the sequences given in the statement of this theorem. Pick a fixed integer k so that $t_k \geq t_0$. Integrating both sides of the Riccati equation from t_k to τ_k we obtain

$$\begin{aligned}
Z(\tau_k) &= Z(t_k) - \int_{t_k}^{\tau_k} Q(t)\Delta t - \int_{t_k}^{\tau_k} F(t)\Delta t \\
&= Z(t_k) - \int_{t_k}^{\tau_k} Q(t)\Delta t - \int_{t_k}^{\sigma(t_k)} F(t)\Delta t - \int_{\sigma(t_k)}^{\tau_k} F(t)\Delta t \\
&= Z(t_k) - F(t_k)\mu(t_k) - \int_{t_k}^{\tau_k} Q(t)\Delta t - \int_{\sigma(t_k)}^{\tau_k} F(t)\Delta t \\
&= Z(t_k) - \mu(t_k)Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}Z(t_k) - \int_{t_k}^{\tau_k} Q(t)\Delta t - \int_{\sigma(t_k)}^{\tau_k} F(t)\Delta t \\
&= Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}[P(t_k) + \mu(t_k)Z(t_k) - \mu(t_k)Z(t_k)] - \int_{t_k}^{\tau_k} Q(t)\Delta t \\
&- \int_{\sigma(t_k)}^{\tau_k} F(t)\Delta t \\
&= Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(t)\Delta t - \int_{\sigma(t_k)}^{\tau_k} F(t)\Delta t. \\
&\leq Z(t_k)[P(t_k) + \mu(t_k)Z(t_k)]^{-1}P(t_k) - \int_{t_k}^{\tau_k} Q(t)\Delta t.
\end{aligned}$$

But using (5) we get that

$$\begin{aligned}
Z(t_k)[P(t_k)+\mu(t_k)Z(t_k)]^{-1}P(t_k) &= Z(t_k)[X(t_k)X^{-1}(\sigma(t_k))P^{-1}(t_k)]P(t_k) \\
&= P(t_k)X^{\Delta}(t_k)X^{-1}(\sigma(t_k)) \\
&= \frac{P(t_k)}{\mu(t_k)}[X^{\sigma}(t_k)) - X(t_k)]X^{-1}(\sigma(t_k)) \\
&= \frac{P(t_k)}{\mu(t_k)} - \frac{1}{\mu(t_k)}P(t_k)X(t_k)X^{-1}(\sigma(t_k)) \\
&< \frac{P(t_k)}{\mu(t_k)}
\end{aligned}$$

where in the last step we used Lemma 4. Hence from the above we have

$$Z(\tau_k) < \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{\tau_k} Q(t)\Delta t.$$

Using $P(\tau_k)+\mu(\tau_k)Z(\tau_k)>0$, we have finally

$$\frac{P(\tau_k)}{\mu(\tau_k)} + \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{\tau_k} Q(t)\Delta t > 0,$$

which is a contradiction. □

Corollary 12 *Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. A necessary condition for the self-adjoint equation $LX(t) = 0$ to be nonoscillatory on $[a,\infty)$ is that for any strictly increasing sequence $\{t_k\} \subset [a,\infty)$ such that $\mu(t_k) > 0$ for $k = 1, 2, \cdots$, with $\lim_{k\to\infty} t_k = \infty$, there is a subsequence $\{t_j\} \subset \{t_k\}$ such that*

$$D_j := \frac{P(t_j)}{\mu(t_j)} + \frac{P(t_{j+1})}{\mu(t_{j+1})} - \int_{t_j}^{t_{j+1}} Q(s)\Delta s > 0$$

for $j = 1, 2, \cdots$.

Proof: We obtain this result by taking $\tau_j = t_{j+1}$ in the previous theorem. □

Corollary 13 *Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Assume that there is a strictly increasing sequence $\{t_k\} \subset [a,\infty)$ such that $\mu(t_k) > 0$ for $k = 1, 2, \cdots$, with $\lim_{k\to\infty} t_k = \infty$. Further assume that there are sequences $\{s_k\} \subset [a,\infty)$ and $\{\tau_k\} \subset [a,\infty)$ such that $\sigma(s_k) > s_k \geq \sigma(\tau_k) > \tau_k \geq \sigma(t_k)$, for $k = 1, 2, \cdots$ such that*

$$\int_{t_k}^{\tau_k} Q(t)\Delta t \geq \frac{P(t_k)}{\mu(t_k)}$$

and

$$\lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)} - \int_{\tau_k}^{s_k} Q(t)\Delta t\right) \leq 0$$

for $k = 1, 2, \cdots$. Then $LX(t) = 0$ is oscillatory on $[a,\infty)$.

Proof: It follows from Weyl's inequality that if A and B are Hermitian matrices, then

$$\lambda_{\min}(A-B) \leq \lambda_{\min}(A) - \lambda_{\min}(B).$$

We will use this fact in the following chain of inequalities. Consider

$$\begin{aligned}
& \lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)} + \frac{P(t_k)}{\mu(t_k)} - \int_{t_k}^{s_k} Q(t)\Delta t\right) \\
= & \lambda_{\min}\left([\frac{P(s_k)}{\mu(s_k)} - \int_{\tau_k}^{s_k} Q(t)\Delta t] - [\int_{t_k}^{\tau_k} Q(t)\Delta t - \frac{P(t_k)}{\mu(t_k)}]\right) \\
\leq & \lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)} - \int_{\tau_k}^{s_k} Q(t)\Delta t\right) - \lambda_{\min}\left(\int_{t_k}^{\tau_k} Q(t)\Delta t - \frac{P(t_k)}{\mu(t_k)}\right) \\
\leq & 0.
\end{aligned}$$

Hence the result follows from Theorem 11. ☐

Corollary 14 *Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Suppose also that*

$$\lim_{t\to\infty} \lambda_{\min}\left(\int_{t_0}^{t} Q(s)\Delta s\right) = \infty \tag{13}$$

and for each $T \in [a,\infty)$ there is a $t \in [T,\infty)$ such that $\mu(t) > 0$ and

$$\lambda_{\min}\left(\frac{P(t)}{\mu(t)} - \int_{T}^{t} Q(s)\Delta s\right) \leq 0 \tag{14}$$

then $LX = 0$ is oscillatory on $[a,\infty)$.

Proof: Let $\{t_k\} \subset [a,\infty)$ be a strictly increasing sequence with $\mu(t_k) > 0$ and $\lim_{k\to\infty} t_k = \infty$. Using (13) we get for each k there is a $\tau_k > \sigma(t_k)$ so that

$$\int_{t_k}^{\tau_k} Q(t)\Delta t \geq \frac{P(t_k)}{\mu(t_k)}$$

$k = 1, 2, \cdots$. Using (14) we get for each $k = 1, 2, \cdots$ there is an $s_k \geq \sigma(\tau_k)$ so that

$$\lambda_{\min}\left(\frac{P(s_k)}{\mu(s_k)} - \int_{\tau_k}^{s_k} Q(t)\Delta t\right) \leq 0.$$

It follows from Corollary 13 that $LX = 0$ is oscillatory on $[a,\infty)$. ☐

Theorem 15 *Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. Assume for each $t_0 \in [a,\infty)$ there is a strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset [t_0,\infty)$ with $\mu(t_k) > 0$ and $\lim_{k\to\infty} t_k = \infty$, and there are constants K_1, K_2 such that $0 < K_1 \leq \mu(t_k) \leq K_2$ for $k = 1, 2, \cdots$, such that*

$$\lim_{k\to\infty} \lambda_{\max}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) \geq \frac{1}{\mu(t_1)}. \tag{15}$$

Further assume that there is a constant M such that

$$\operatorname{tr}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) \geq M \tag{16}$$

for $k \geq 1$. Then equation (6) *is oscillatory on* $[a, \infty)$.

Proof: Assume not, then equation (6) is nonoscillatory on $[a, \infty)$. This implies that there is a nontrivial prepared solution $X(t)$ of (6) and a $t_0 \in [a, \infty)$ such that

$$X^*(\sigma(t)X(t) > 0$$

on $[t_0, \infty)$. We make the Riccati substitution

$$Z(t) = X^{\Delta}(t)X^{-1}(t)$$

for $t \in [t_0, \infty)$, then by Theorem 5 we get that

$$I + \mu(t)Z(t) > 0$$

on $[t_0, \infty)$ and $Z(t)$ is a Hermitian solution of the Riccati equation

$$Z^{\Delta}(t) + Q(t) + F(t) = 0$$

on $[t_0, \infty)$, where $F := Z^*(P + \mu Z)^{-1}Z$. Corresponding to t_0 let $\{t_k\}_{k=1}^{\infty} \subset [t_0, \infty)$ be the sequence guaranteed in the statement of this theorem. Integrating both sides of the Riccati equation from t_1 to t_k where $k > 1$ gives

$$\begin{aligned}
Z(t_k) + \int_{t_1}^{t_k} Q(t)\Delta t + \int_{t_1}^{t_k} F(t)\Delta t &= Z(t_1) \\
Z(t_k) + \int_{t_1}^{t_k} Q(t)\Delta t + \int_{t_1}^{\sigma(t_1)} F(t)\Delta t + \int_{\sigma(t_1)}^{t_k} F(t)\Delta t &= Z(t_1) \\
Z(t_k) + \int_{t_1}^{t_k} Q(t)\Delta t + F(t_1)\mu(t_1) + \int_{\sigma(t_1)}^{t_k} F(t)\Delta t &= Z(t_1).
\end{aligned}$$

Simplifying we get

$$Z(t_k) + \int_{t_1}^{t_k} Q(t)\Delta t + \int_{\sigma(t_1)}^{t_k} F(t)\Delta t = Z(t_1)\left(I + \mu(t_1)Z(t_k)\right)^{-1}.$$

Hence

$$\begin{aligned}
&\lambda_{\max}\left(Z(t_k) + \int_{t_1}^{t_k} Q(t)\Delta t + \int_{\sigma(t_1)}^{t_k} F(t)\Delta t\right) \\
&= \lambda_{\max}\left(Z(t_1)\left(I + \mu(t_1)Z(t_1)\right)^{-1}\right).
\end{aligned} \tag{17}$$

By Weyl's inequality

$$\begin{aligned}
&\lambda_{\max}\left(Z(t_1)\left(I+\mu(t_1)Z(t_1)\right)^{-1}\right)\\
&> \lambda_{\min}\left(Z(t_k)\right)+\lambda_{\max}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right)+\lambda_{\min}\left(\int_{\sigma(t_1)}^{t_k} F(t)\Delta t\right)\\
&\geq \lambda_{\min}\left(Z(t_k)\right)+\lambda_{\max}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right). \qquad (18)
\end{aligned}$$

Taking the limit of both sides as $k \to \infty$ and using

$$\lim_{k\to\infty} \lambda_{\min}\left(Z(t_k)\right) = 0, \qquad (19)$$

which we will prove later, we get

$$\begin{aligned}
&\lambda_{\max}\left(Z(t_1)\left(I+\mu(t_1)Z(t_1)\right)^{-1}\right)\\
&\geq \lim_{k\to\infty}\lambda_{\max}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right)\\
&\geq \frac{1}{\mu(t_1)}
\end{aligned}$$

But in Theorem 7 we showed that $I+\mu(t_1)Z(t_1) > 0$ implies that

$$\lambda_{\max}\left(Z(t_1)\left(I+\mu(t_1)Z(t_1)\right)^{-1}\right) < \frac{1}{\mu(t_1)} \qquad (20)$$

and this gives us a contradiction. Hence to complete the proof of this theorem it remains to prove that (19) holds. In fact, we shall prove that

$$\lim_{k\to\infty} \lambda_i\left(Z(t_k)\right) = 0$$

for $1 \leq i \leq n$, which includes (19) as a special case. To do this we first show that

$$\lim_{k\to\infty} \lambda_i\left(F(t_k)\right) = 0$$

for $1 \leq i \leq n$. Since $F(t) \geq 0$ implies $\operatorname{tr}(F(t)) > 0$ we have

$$\begin{aligned}
\sum_{j=1}^{k}\mu(t_j)\lambda_i\left(F(t_j)\right) &\leq \sum_{j=1}^{k}\mu(t_j)tr\left(F(t_j)\right)\\
&= \sum_{j=1}^{k}\int_{t_j}^{\sigma(t_j)} tr\left(F(t)\right)\Delta t\\
&\leq \int_{t_1}^{\sigma(t_k)} \operatorname{tr}\left(F(t)\right)\Delta t\\
&\leq tr\left(\int_{t_1}^{\sigma(t_k)} F(t)\Delta t\right)\\
&\leq n\lambda_{\max}\left(\int_{t_1}^{\sigma(t_k)} F(t)\Delta t\right)
\end{aligned}$$

for all $k > 1$. We now show that the sequence

$$\lambda_{\max}\left(\int_{t_1}^{t_k} F(t)\Delta t\right),$$

$k \geq 1$ is bounded. From (18) and (20) we get

$$\frac{1}{\mu(t_1)} > \lambda_{\min}\left(Z(t_k)\right) + \lambda_{\max}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right)$$

Using (17) and (20) and applying Weyl's inequality twice yields

$$\begin{aligned}\frac{1}{\mu(t_1)} &> \lambda_{\max}\left(Z(t_k) + \int_{t_1}^{t_k} Q(t)\Delta t + \int_{\sigma(t_1)}^{t_k} F(t)\Delta t\right)\\ &\geq \lambda_{\min}\left(Z(t_k)\right) + \lambda_{\min}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) + \lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(t)\Delta t\right).\end{aligned}$$

It follows that

$$\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)} > \lambda_{\min}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) + \lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(t)\Delta t\right). \tag{21}$$

From Theorem 7, we can, without loss of generality, assume that

$$\lambda_{\max}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) < \frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)} \tag{22}$$

holds for $k > 1$.

Therefore

$$\begin{aligned}M &\leq tr\left(\int_{t_1}^{t_k} Q(t)\Delta t\right)\\ &= \sum_{i=1}^{n} \lambda_i\left(\int_{t_1}^{t_k} Q(t)\Delta t\right)\\ &= \lambda_{\min}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) + \sum_{i=1}^{n-1} \lambda_i\left(\int_{t_1}^{t_m} Q(t)\Delta t\right)\\ &< \lambda_{\min}\left(\int_{t_1}^{t_k} Q(t)\Delta t\right) + (n-1)\left(\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}\right).\end{aligned}$$

Solving for the last term on the right hand side of the inequality (21) and using the above inequality we get

$$\lambda_{\max}\left(\int_{\sigma(t_1)}^{t_k} F(t)\Delta t\right) < n\left(\frac{1}{\mu(t_1)} + \frac{1}{\mu(t_k)}\right) - M$$

Therefore

$$\sum_{j=1}^{\infty} \mu(t_j) tr\left(F(t_j)\right) < \infty$$

and so since

$$\lambda_i\left(F(t)\right) \leq tr\left(F(t)\right)$$

and $0 < K_1 \leq \mu(t_k) \leq K_2$ for $k = 1, 2, \cdots$ it follows that

$$\lim_{k\to\infty} \lambda_i\left(F(t_k)\right) = 0.$$

That is,

$$\lim_{k\to\infty} \frac{\left(\lambda_i[Z(t_k)]\right)^2}{1+\mu(t_k)\lambda_i[Z(t_k)]} = 0$$

for $i = 1, 2, \cdots, n$. Similar to the argument to prove (9) in Theorem 7 we can show

$$\lambda_i\left(F(t_1)\right) = \frac{d_i^2}{1+\mu(t_1)d_i}. \tag{23}$$

Therefore,

$$0 \leq \frac{\left(\lambda_i[Z(t_k)]\right)^2}{1+K_2\lambda_i[Z(t_k)]} \leq \frac{\left(\lambda_i[Z(t_k)]\right)^2}{1+\mu(t_k)\lambda_i[Z(t_k)]}$$

which implies

$$\lim_{k\to\infty} \frac{\left(\lambda_i[Z(t_k)]\right)^2}{1+K_2\lambda_i[Z(t_k)]} = 0$$

for $i = 1, 2, \cdots, n$, which in turn implies that

$$\lim_{k\to\infty} \lambda_i[Z(t_k)] = 0$$

for $i = 1, 2, \cdots, n$. This completes the proof. □

Applying Theorem 15 one gets the following example.

Example 16 *If* $\mathbb{T} = hN_0$, *where* $h > 0$, $c \geq \frac{1}{h^2}$, *and the function* q *has the values*

$$c,\ -\frac{c}{2},\ -\frac{c}{2},\ \frac{c}{3},\ \frac{c}{3},\ \frac{c}{3},\ -\frac{c}{4},\ -\frac{c}{4},\ -\frac{c}{4},\ -\frac{c}{4},\ \frac{c}{5}, \cdots$$

at $0, h, 2h, 3h, \cdots$ *respectively, then* $x^{\Delta\Delta} + q(t)x^{\sigma} = 0$ *is oscillatory on* $\mathbb{T}$.

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